

# **Principles Of Mechanics**

**John. L Synge**

# PRINCIPLES *of* MECHANICS

BY

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Le savant doit ordonner; on fait la science avec des faits comme une maison avec des pierres, mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.

**HENRI POINCARÉ**

# PREFACE TO THE FIRST EDITION

In a sense this is a book for the beginner in mechanics, but in another sense it is not. From the time we make our first movements, crude ideas on force, mass, and motion take shape in our minds. This body of ideas might be reduced to some order at high school (as crude ideas of geometry are reduced to order), but that is not the educational practice in North America. There is rather an accumulation of miscellaneous facts bearing on mechanics, some mathematical and some experimental, until a state is reached where the student is in danger of being repelled by the subject, as a chaotic jumble which is neither mathematics nor physics.

This book is intended primarily for students at this stage. The authors' ambition is to reveal mechanics as an orderly self-contained subject. It may not be quite so logically clear as pure mathematics, but it stands out as a model of clarity among all the theories of deductive science.

The art of teaching consists largely in isolating difficulties and overcoming them one by one, without losing sight of the main problem while attending to the details. In mechanics, the main problem is the problem of equilibrium or motion under given forces—the details are such things as the vector notation, the kinematics of a rigid body, or the theory of moments of inertia. If we rush straight at the main problem, we become entangled in the details and have to retrace our steps in order to deal with them. If, on the other hand, we decide to settle all details first, we are apt to find them uninteresting because we do not see their connection with the main problem. A compromise is necessary, and in this book the compromise consists of the division into Plane Mechanics ([Part I](#)) and Mechanics in Space ([Part II](#)). These titles must, however, be regarded only as rough indications of the contents. [Part I](#) includes some of the easier portions of three-dimensional theory, while [Part II](#) contains an introduction to the special theory of relativity, with mechanics in only one spatial dimension!

There is, of course, nothing novel in regarding plane mechanics as the preliminary field; but it is rather unusual to divide the subject in this way in a single volume, or even in a sequence of volumes. It has made the

task of writing more difficult, but the authors have felt it worth while. Many of the most interesting results in statics and dynamics belong to the plane theory, and it is unfair to deny the reader access to them until he has mastered the more elaborate technique required for three dimensions.

**Part I** is complete in itself and might be used as a textbook in plane statics and dynamics, with some excursions into three-dimensional theory. Vector notation is introduced, but used sparingly. The reader should have a fair knowledge of calculus, elementary differential equations, and some analytical geometry. Practical experience in physics is not essential but very desirable; mechanics is at root a physical subject and should not be treated merely as an excuse for the exercise of mathematical techniques.

In **Part II** the language of vectors is used extensively. A knowledge of three-dimensional analytical geometry is required and greater power in the use of mathematical processes. This part is complete in itself, except for occasional references to **Part I**. The selection of particular applications follows conventional lines, except for one novel feature—a section on electron optics. Chapters on Lagrange's equations and on the special theory of relativity are included.

The book has developed from lectures delivered by both authors to Honor Students in their second and third years at the University of Toronto. These lectures cover about 110 periods of 50 minutes, and it has been found that the work can be done fairly adequately in that time. But this does not allow sufficiently for the working of problems with the classes; it is felt that 150 periods might well be spent on the contents of the book, were it not for other demands on the students' time.

Each chapter is followed by a summary. The summaries to the chapters dealing with methods are naturally the more fundamental—there is little hope of being able to attack problems unless one is thoroughly familiar with the general principles outlined there. On the other hand, the summaries to the chapters dealing with applications are intended to provide only a synopsis of what has been done.

Many of the exercises are taken with permission from examination papers set in the University of Toronto and printed by the University Press. In each set of exercises, the first few problems are so simple that

failure to solve them will reveal a lack of understanding of basic methods, rather than a deficiency in skill and ingenuity.

The equations are numbered in such a way that, when read as decimals, they stand in their proper order. The integer represents the chapter, the first decimal place represents the section, and the last two decimal places the position of the equation in the section.

Debts to other textbooks are too numerous to acknowledge. But we would like to pay tribute to two books and recommend them to the reader who wishes to pursue the subject further. They are E. T. Whittaker's *Analytical Dynamics* (Cambridge University Press) and P. Appell's *Mécanique rationnelle* (Gauthier-Villars). These books have suggested the possibility of reconciling in a textbook on mechanics two opposing goals—the reduction of the subject to a compact and classified form and its exposition with sufficient fullness to make the arguments easy to follow.

We gratefully acknowledge assistance and advice received from our colleagues, Professor H. S. M. Coxeter, Professor A. F. Stevenson, Dr. A. Weinstein, and Mr. A. W. Walker. We are under a particular debt to Professor L. Infeld, who read most of the manuscript and has been unsparing in frank criticism and suggestions; if we have succeeded in avoiding dullness and obscurity, it is due in no small measure to him.

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**PART I**  
**PLANE MECHANICS**

# CHAPTER I

## FOUNDATIONS OF MECHANICS

### 1.1. SOME PHILOSOPHICAL IDEAS

Why do we study mechanics? There are at least three reasons. First, we live in an age of machinery, which cannot be designed without a knowledge of mechanics; in fact, it is the most fundamental subject in engineering. Secondly, mechanics plays a basic part in physics and astronomy, contributing to our knowledge of the working of nature. Thirdly, the mathematician is interested in mechanics, both in the logic of its foundations and in the methods employed; a considerable portion of mathematics was developed for the express purpose of solving mechanical problems.

The subject of mechanics is not a mere collection of facts. From certain simple hypotheses an elaborate theory is built up. Anyone who has studied the subject should be able to answer questions of interest to engineers and physicists; that is to say, he should be able to *apply* his knowledge. But he should also have a fair idea of the logical structure. A successful textbook has to steer a middle course between undue concentration on the mere working out of problems on the one hand, and an over-elaborate development of logical structure on the other.

### **The two ways of thinking.**

What the student of mechanics requires more than anything else is the development of a certain point of view which is difficult to describe in a few words. Since the reader is expected to have a fair knowledge of geometry, it will be helpful to consider the ways in which we think about that subject.

Every student of geometry learns to think in two ways. First, there is the *physical* way, in which a point is a small dot on a sheet of paper, a

the bar is it essential for us to bear in mind, and which may we neglect as incidental? Can we predict the periodic time of oscillation without knowing the material of which the bar is constructed? Does the form of the cross section of the bar matter? Does it make any difference whether the bar is supported on a knife-edge or by bearings? The cautious well-informed physicist would say that all these things mattered and many others. One material yields more than another, the form of cross section influences the distribution of material, and a change in the mode of suspension may alter the axis about which the pendulum oscillates. But if we were as cautious as this we should have no science of mechanics. To start on the problem, at any rate, we must simplify it ruthlessly. So we think of the bar as a rigid mathematical straight line and the support as a fixed mathematical point. Now we have a problem which is reasonably simple to handle mathematically. Strictly speaking, no properties are incidental. Even the color of the bar affects the pressure of light on it; a subway train stopping five hundred miles away may cause a vibration in the support and affect the motion of the bar. Common sense, which is the accumulated experience of centuries, gives us some guide as to the factors which we may neglect.

## **Mathematical models.**

Gradually stripping physical things of attributes which are unimportant for the question in hand, we arrive at a mathematical way of thinking about nature. The particular mathematical model\* to be used on a given occasion depends on that occasion. Consider the earth, for example. The simplest model of the earth is a particle, a mathematical point with mass. This model suffices to obtain the earth's orbit round the sun, but obviously will not do for the discussion of tides or lunar eclipses. For these phenomena we may think of the earth as a rigid sphere, but this model will not serve for the discussion of the precession of the equinoxes (for which we require an ellipsoidal rigid body) or for the discussion of earthquakes (for which we require an elastic sphere). Thus there are many mathematical models for the earth, and the one which we choose depends on the question we are discussing at the moment.

Step (3) belongs largely to pure mathematics, requiring no particular

knowledge of, or interest in, the physical problem. Nevertheless, it is often of the greatest assistance to the mathematician to bear the physical problem continually in mind; in this way, methods of attack may be suggested to him.

The fourth step in general presents no difficulty, provided that we are clear as to the things in nature which correspond to the things in our mathematical model.

The technical details of the fifth step belong to experimental physics or observational astronomy, and with them we shall not be concerned. But we are interested in the fact that the conclusions drawn from a mathematical theory are, or are not, physically true, within the limits of accuracy of observation.

It is necessary to distinguish between *mathematical* truth and *physical* truth. In developing the theory of mechanics, we shall try to make the mathematical arguments fairly complete, so that we can have confidence that the conclusions follow logically from the hypotheses, i.e., that they are mathematically true. We should not undertake this work, however, if we had not confidence that our conclusions are also physically true, in the sense that they agree with observation. A vast accumulation of physical results confirms our confidence. Nevertheless, it would be too much to claim that *all* our conclusions are physically valid.

Attempts to construct a successful model of an atom on the basis of Newtonian mechanics have failed. This failure led to the invention of quantum mechanics. We may say in general that Newtonian models of small-scale phenomena have not been successful, whereas at the other end of the scale we find difficulty also in the large-scale phenomena of astronomy. In spite of the many triumphs of Newtonian mechanics in dynamical astronomy, there remain a few phenomena which are in apparent disagreement with it; the best-known concerns the orbit of the planet Mercury. This difficulty was overcome when Einstein created the general theory of relativity.

To explore with any degree of completeness the theories referred to above would demand a course of study far wider than that covered in this book. The reader may feel disappointed that at this stage he cannot reach the forefront of our mechanical knowledge. To encourage him, however, it may be pointed out that as long as the physical problems concern only



apparatus of an intermediate scale, i.e., neither atomic on the one hand nor astronomical on the other, one may have complete confidence that no experimental technique can reveal any discrepancy between observation and the conclusions drawn from Newtonian mechanics. This confidence may even be extended to astronomy, because there the relativistic effects are extremely minute; the vast body of calculations of dynamical astronomy are still safely based on Newtonian mechanics.

Relativity and quantum mechanics not only enable us to obtain results which are physically true—they also throw light on such basic philosophical ideas as simultaneity and causality. [Chapter XVI](#) contains an introduction to the special theory of relativity. The general theory of relativity and quantum mechanics both lie outside the scope of this book.

## 1.2. THE INGREDIENTS OF MECHANICS

In any subject there are words which occur again and again, like the words “point,” “line,” and “circle” in elementary geometry. As well as these technical words, there occur ordinary words with the meanings of which we are supposed to be familiar. When we start a new subject, we are not expected to know what the technical words mean. They are introduced with some formality, being in fact given definitions. A definition is itself only a set of words and may not mean much; the general idea is to explain a new thing in terms of things already familiar.

We are now to try to create mathematical models of physical things. We start with a fair general unprecise knowledge of the world around us; the places in our minds reserved for the mathematical models are supposed to be absolutely blank. If we opened these places for the actual world to rush in, we should be overwhelmed with confusion. We guard the door and admit only a few ingredients of simple mathematical character.

### Particles.

The first thing we admit is a *particle*. We have seen tiny scraps of matter and it is not difficult for us, with our training in geometry, to think of a scrap of matter with no size at all, but with a definite position; that is a particle. When we have to deal with a physical problem in which a body is very small in comparison with distances or lengths involved (for

example, the earth in comparison with its distance from the sun, or the bob of a pendulum in comparison with the string), we may represent that body in our mathematical model by a particle.

## Mass.

Primitive trade was a matter of barter; later, money was introduced as a standard scale for comparison of values, and equivalence in value is now expressed by equality of price. This exemplifies a process of deep importance in science, namely, a concentration on some characteristic (value) of a thing and its expression by means of a number (price). A barrel of apples is very different from a pair of shoes, but they may be equivalent if value is the only characteristic in which we are interested. That the price is the same expresses complete equivalence as far as our purse is concerned.

Consider now a great variety of bodies—pieces of stone, iron, gold, wood, etc.—and mechanical experiments performed on them. As examples, we mention two experiments:

(i) The body is placed in the pan of a spring balance and the reading noted.

(ii) The body is fired from a gun by means of a definite explosive charge and passes into a block of wood, the resulting displacement of which is noted.

If  $A$  and  $B$  are two pieces of iron, as nearly identical in shape and size as it is possible to make them, they will of course give the same results when used in any experiment, performed first using  $A$  and then repeated using  $B$  instead. But it is a remarkable fact, resting on long experience, that two bodies  $A$  and  $B$  may differ in material, size, shape, etc., and yet give the same result in a great variety of mechanical experiments. We then say that they are *mechanically equivalent*. A piece of wood and a piece of gold may be mechanically equivalent, just as a barrel of apples and a pair of shoes may be equivalent in value.

As we assign a price to each article of trade, so we may assign a number to each piece of matter, equality of these numbers implying mechanical equivalence. This number is called *mass* and is usually denoted by  $m$ . Following the analogy of money, based on a standard substance (gold), it is easy to see how a scale of mass is to be

constructed. We start with a number of identical pieces of some standard material such as platinum, and we assign to the mass of each the value unity ( $m = 1$ ). When  $n$  of these pieces are lumped together, we assign to the mass of the lump the value  $n(m = n)$ . By cutting the pieces, we can construct bodies with fractional masses and so obtain a set of standard bodies of all possible masses. Then, to assign a mass to a body  $A$  (not of the standard material), we subject it to experiments and find that standard body  $B$  to which it is mechanically equivalent. We then say that the mass of  $A$  is the same as the mass of  $B$ .

The comparison of masses is usually made by weighing as in the experiment (i) mentioned above, except that for reasons of accuracy the spring balance is replaced by a laboratory balance. Thus, in practice, two bodies are said to have the same mass when they have the same weight.

The above considerations deal with physical bodies. In the mathematical model in which these bodies are represented by particles, we are to regard each particle as having attached to it a positive number  $m$ , its mass, which does not change during the history of the particle.

In dealing with a system of particles, we define the mass of the system to be the sum of the masses of the particles which compose it.

## **Rigid bodies.**

We have now admitted as a mathematical model the particle with mass. The next thing to consider is the *rigid body*.

It is a matter of common experience that bodies may be soft like rubber or hard like steel. Even the hardest body, however, changes its size and shape by measurable amounts under the action of sufficiently great forces. But just as we idealized the small body of our experience into the particle with position but no size, so we idealize the hard body of our experience into the rigid body, which never undergoes any change of size or shape. The rigid body is now admitted as a mathematical model.

We pause for a moment to examine critically something written just above. We spoke of a body changing its size and shape. What does this really mean? Suppose, for example, we have a bar of steel with two marks on it. Alongside the bar we lay a graduated measuring scale and note the readings on the scale opposite the two marks on the bar. Then we pull the ends of the bar and note the readings again. The difference

between them is greater than it was before; hence we say that the bar has increased in length.

However, an argumentative person might assert that this was an incorrect statement; he might hold that the length of the bar was the same as before but that the measuring scale had shrunk. We cannot say that he is wrong in taking this point of view until we clarify our ideas as to the meaning of the word “length.”

The idea of length is one that involves the comparison of two bodies. We decide once for all on a unit of length by making two marks on a piece of metal and stating conditions with regard to temperature and pressure under which measurements are to be made with this piece of metal. We were perhaps a little hasty in admitting a rigid body as a mathematical model, because there is no sense in talking about a single rigid body; we must have some means of measuring it and testing that it is rigid. So when we admit the rigid body, we shall at the same time admit a *measuring scale*. When we say that a body is rigid, we mean that measurements of distances between marks on it always have the same values, the measurements being made with the measuring scale.

## **Events.**

The word *event* is familiar in ordinary speech. It usually denotes something a little out of the ordinary, something that occurs in a fairly limited region of space and is of fairly short duration. Thus a football game or the arrival of a train might be described as an event. The word has now acquired an idealized scientific meaning, the idealization involved being rather similar to that by which we created the concept of particle. Instead of occupying a fairly limited region in space, an event (in our mathematical model) occurs at a mathematical point; and instead of being of fairly short duration, it occurs instantaneously. We do not carry over into our mathematical model the slightly dramatic meaning attached to the word in ordinary life. Anything that happens may be called an event. Even the continued existence of a particle forms a series of events.

## **Frames of reference.**

In describing an event in ordinary life, it is usual to specify the place



and time. Thus it is recorded of the sinking of a ship that it occurred at a certain latitude and longitude, and at a certain Greenwich mean time. Latitude and longitude define position on the earth's surface; we are here using the earth as a *frame of reference*. This is the most familiar frame of reference, but others may be used. Astronomers prefer a frame of reference in which the sun is fixed and which does not share in the earth's motion of rotation. Also, the interior of a train, streetcar, elevator, or airplane may be used. The essential thing about a frame of reference is that it should be fairly rigid.

In our mathematical model, we employ a rigid body as frame of reference. As we may introduce any number of rigid bodies moving relative to one another, we have thus at our disposal any number of frames of reference. Selecting one of these and taking rectangular axes of coordinates in it, we assign to any event a set of three numbers  $x, y, z$ , the coordinates in the frame of reference of the point where the event occurs.

## Time.

An event has not only position; it also has a time of occurrence. This we have now to consider.

The possibility of repeating an experiment forms the basis of experimental science. It is assumed that, if an experiment is repeated under the same conditions, the same results will be obtained. Consider, for example, a tank of water drained through a hole in the bottom, and then refilled and drained again. Strictly speaking, it is impossible to reproduce conditions exactly, and we have to use judgment to decide whether the new conditions are sufficiently near the old. But in an ideal sense we may think of an experiment repeated over and over again under exactly the same conditions.

To define *time*, we think of some experiment which can be repeated over and over again, a new experiment starting just when the preceding one ends. Denoting time by  $t$ , we assign the value  $t = 0$  to the beginning of the first experiment,  $t = 1$  to the beginning of the second experiment,  $t = 2$  to the beginning of the third experiment, and so on. The repeated experiment thus forms a *clock* for the measurement of time; we shall call the unit of time given by some such ideal experiment a *Newtonian unit*. This is the procedure actually adopted in practice. In a watch, the

greater forces due to the pressure of steam, the weight of water, the explosive pressure of gasoline, or forces of electromagnetic type. One also admits the existence of huge forces beyond human control, such as the gravitational attraction exerted by the sun on the earth.

On the basis of our experience with simple muscular forces, we think of the idealized force of our mathematical model as something which has

- (i) a point of application,
- (ii) a direction,
- (iii) a magnitude.

For the development of general results in theoretical mechanics, it would be sufficient to represent the magnitude of a force by a letter, standing for some unspecified numerical value. But when we wish to make predictions regarding a physical system subject to forces, we require a definite procedure by means of which we may assign numerical values to their magnitudes. We must, in fact, define a *unit of force*.

There has been some controversy about this question. Though all are agreed as to the form of theory which we should ultimately obtain, there has been disagreement regarding the proper order of introduction of the various parts of the theory. Thus we might assume a statement  $A$  as an axiom and deduce a statement  $B$  from it, or alternately we might assume  $B$  and deduce  $A$ . The order of presentation chosen in this book seems to the authors the most natural; but it is hoped that the reader will explore for himself the possibility of a different approach.\*

We define the unit of force in terms of a stretched spring; it is that force which produces some standard extension in some standard spring. Later we shall link up the unit of force with the units of mass, length, and time; but for the present the unit is to be regarded as arbitrary.

To measure a force, we examine the extension which it produces in a battery of standard springs side by side, all identical with one another. If the standard extension is produced in  $n$  springs, then the force is of magnitude  $n$ . If the magnitude of the force in question is not an integer, we reproduce it  $m$  times so as to get the standard extension in some number  $n$  of standard springs; then the magnitude of the force is  $n/m$ .

Having thus given a means of measuring force, we may construct a simplified apparatus. Taking any spring, fixed at one end, we mark its extensions under the action of measured forces. In this way, we *calibrate*

the spring; the calibrated spring may be used directly for the measurement of a force.

We have preferred to use the tension in a spring rather than the weight of a body as the foundation of our definition of force. It is customary for many practical purposes to speak of a force of so many pounds weight (lb. wt.); by a force of 10 lb. wt., we mean a force equal to the weight of a mass of 10 lb. Although this practice is convenient and adequate for many purposes, it is open to objection on the following ground: If the weight of a body is measured by means of a calibrated spring at two different latitudes, the results are not the same (see Sec. 5.3). A definition of force based on the extension of a spring gives a consistent theory without contradictions, whereas a definition based on weight would involve us in explanations as to why a spring, showing the same definite extension in Toronto and in Panama, should exert different forces in the two places.

In describing particles, rigid bodies, and forces we have introduced the basic ingredients of mechanics. As we proceed, other ingredients will appear, but it is remarkable how much of the subject turns on the simple concepts just mentioned.

### 1.3. INTRODUCTION TO VECTORS. VELOCITY AND ACCELERATION

#### Definition of a vector.

In order to reproduce a game of chess, we must be able to describe the moves. There is, of course, an accepted way of doing this, but we shall describe another. Let letters  $A, B, C, \dots$  (supplemented with other symbols to make up 64) be assigned to the squares of the board, one letter to each square. Then symbols such as  $\overrightarrow{AB}, \overrightarrow{CF}, \overrightarrow{UB}$  will represent definite moves, the symbol  $\overrightarrow{AB}$ , for example, meaning that a piece is moved from the square  $A$  to the square  $B$ .

More generally, if we carry a particle from a position  $A$  to a position  $B$  in space, the operation which we perform may be represented symbolically by  $\overrightarrow{AB}$ . The directed segment drawn from  $A$  to  $B$ , or the carrying operation—or indeed any physical quantity which can be represented by the directed segment—is called a *vector*,\* and the symbol

$\vec{AB}$  is used for any of them.

A vector  $\vec{AB}$  has the following characteristics:

- (i) an origin or point of application ( $A$ );
- (ii) a direction (defined analytically by the three direction cosines of  $\vec{AB}$  with respect to rectangular axes);
- (iii) a magnitude (the length  $AB$ ).

A number of fundamental physical quantities have these characteristics—for example, a force or the velocity of a particle; each of these quantities may be represented by a directed segment and is therefore a vector. They are to be distinguished from quantities such as mass or kinetic energy, which do not involve the idea of direction and are described each by a single number. Quantities of this latter type are called *scalars*.

It is convenient to employ the word “vector” in a slightly wider sense than that given above and to define the following :

- (i) free vector;
- (ii) sliding vector;
- (iii) bound vector.

A *free vector* is any one of a system of directed segments having a common direction and magnitude but different origins. A physical quantity equally well represented by any one of such directed segments is also called a free vector. Such, for example, is the displacement, without rotation, of a rigid body, which is equally well represented by any one of the directed segments giving the displacements of its various points.

A *sliding vector* is any one of a system of directed segments obtained by sliding a directed segment along its line. A physical quantity equally well represented by any one of such directed segments is also called a sliding vector. Such, for example, is a force acting on a rigid body, which (by the principle of the transmissibility of force proved on page 64) may equally well be applied at any point on its line of action.

A *bound vector* is a unique directed segment, or a physical quantity so represented. Such, for example, is a force acting on an elastic body; we cannot in general alter this force, by any displacement of the directed segment representing it, without changing its effect.



By the word “vector,” without an adjective, we shall generally understand “free vector”; but where we have to speak of bound or sliding vectors, it will be unnecessary to use the qualifying adjective when it may be understood from the context. Throughout the rest of this section, the vectors are to be regarded as free.

## Notation.

A vector is indicated in print by a boldface letter ( $\mathbf{P}$ ): in manuscript work the symbol may be underlined ( $\underline{P}$ ) or an arrow may be written on top ( $\overrightarrow{P}$ ). Its magnitude is denoted by the same letter in ordinary type, or by an unmarked symbol in manuscript work ( $P$ ). A vector of unit magnitude is called a *unit vector*. To refer to a bound or sliding vector, we may write “ $\mathbf{P}$  acting at the point  $A$ ” or “ $\mathbf{P}$  acting on the line  $L$ ,” if there is any doubt as to the origin or line.

Two vectors are equal to one another when they may be represented by equal parallel directed segments with the same sense. We use the usual sign of equality and write

$$\mathbf{P} = \mathbf{Q}.$$

The sign of equality carries the usual algebraic property: vectors equal to the same vector are equal to one another.

## Multiplication of a vector by a scalar.

Let  $\mathbf{P}$  be a vector and  $m$  a scalar. We define the product of  $m$  and  $\mathbf{P}$  (written  $m\mathbf{P}$  or  $\mathbf{P}m$ ) as follows: If  $m$  is positive, then  $m\mathbf{P}$  has the same direction as  $\mathbf{P}$  and a magnitude  $mP$ ; if  $m$  is negative, then  $m\mathbf{P}$  has a direction opposite to that of  $\mathbf{P}$  and a magnitude  $-mP$ .

We write  $(-1)\mathbf{P} = -\mathbf{P}$ ; thus  $-\mathbf{P}$  is the vector  $\mathbf{P}$  reversed.

## Addition of vectors.

The sum of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is written  $\mathbf{P} + \mathbf{Q}$ ; it is defined as the vector represented by the diagonal  $\overrightarrow{AD}$  of a parallelogram of which two adjacent sides  $\overrightarrow{AB}$ ,  $\overrightarrow{AC}$  represent  $\mathbf{P}$  and  $\mathbf{Q}$ , respectively (Fig. 1). Obviously, an alternative way of constructing  $\mathbf{P} + \mathbf{Q}$  is the following

a straight line. But that does not prevent us from applying the above definition, regarded as a limiting process. It is easily seen that, if  $\mathbf{P}$  and  $\mathbf{Q}$  have the same direction, then  $\mathbf{P} + \mathbf{Q}$  has also that direction and a magnitude  $P + Q$ ; if they have opposite directions and  $\mathbf{P}$  is the greater, then  $\mathbf{P} + \mathbf{Q}$  has the direction of  $\mathbf{P}$  and a magnitude  $P - Q$ . Comparing this with the definition of the product of a vector by a scalar, we find in particular that

$$\mathbf{P} + \mathbf{P} = 2\mathbf{P},$$

and we verify generally that the multiplication of a vector by a scalar is *distributive* both with respect to the scalar and to the vector; this means that we have

$$(1.301) \quad (m + n)\mathbf{P} = m\mathbf{P} + n\mathbf{P},$$

$$(1.302) \quad m(\mathbf{P} + \mathbf{Q}) = m\mathbf{P} + m\mathbf{Q}.$$

It is an immediate consequence of the definition that the addition of vectors is *commutative*, that is,

$$\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}.$$

The subtraction of vectors is immediately effected by writing

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q})$$

and applying the rule for addition. The difference between  $\mathbf{P}$  and  $\mathbf{Q}$  is easily constructed as follows:

Draw  $\vec{AB}$ ,  $\vec{AC}$  to represent  $\mathbf{P}$ ,  $\mathbf{Q}$ , respectively (Fig. 3); then  $\vec{CB}$  represents  $\mathbf{P} - \mathbf{Q}$ .

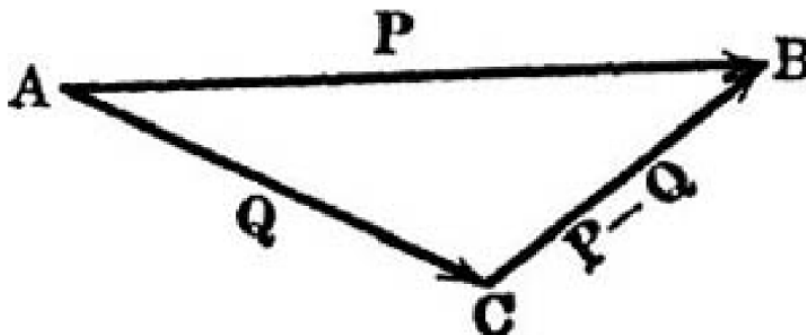


FIG. 3.—Subtraction of vectors.

Applying the rule for subtraction to the case  $\mathbf{P} - \mathbf{P}$ , we obtain a vector of zero magnitude, which we denote by  $\mathbf{0}$ , so that

$$\mathbf{P} - \mathbf{P} = \mathbf{0}.$$

We call  $\mathbf{0}$  the *zero vector*; all vectors of zero magnitude are regarded as equal to one another.

Any unfamiliar symbol containing vectors must be approached with caution. It may mean nothing at all (for example, we never attempt to define the sum of a scalar and a vector,  $m + \mathbf{P}$ ); on the other hand, it may be given a meaning. On the basis of previous definitions,  $\mathbf{P} + \mathbf{Q} + \mathbf{R}$  has no meaning, because we have defined the sum of *two* vectors, not *three*. But

$$(\mathbf{P} + \mathbf{Q}) + \mathbf{R}$$

has a meaning, if we regard the parentheses as carrying the instruction to add  $\mathbf{P}$  and  $\mathbf{Q}$ , and then add  $\mathbf{R}$  to that sum;

$$\mathbf{P} + (\mathbf{Q} + \mathbf{R})$$

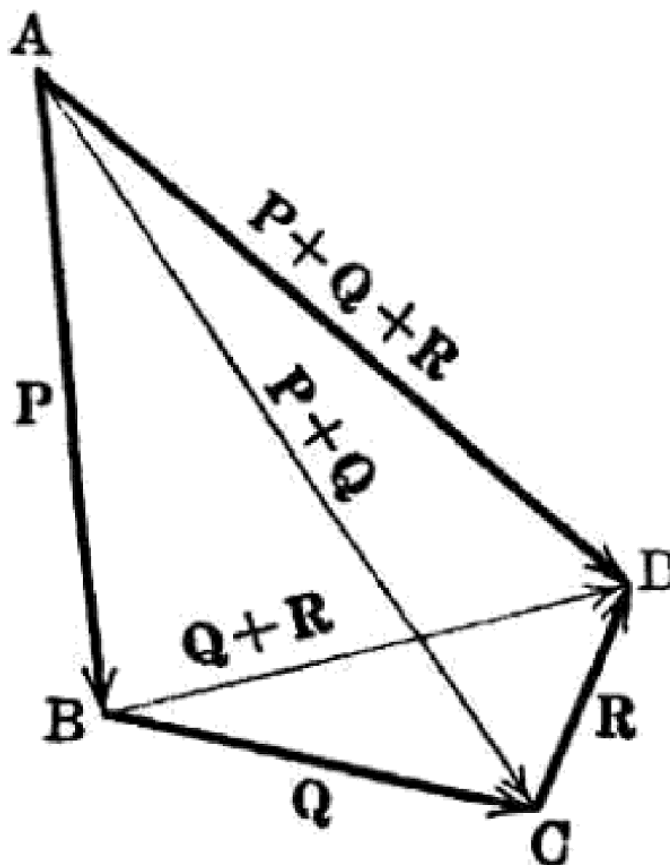


FIG. 4.—The associative property of vector addition.

has a meaning, also. It is then easy to see that

$$(\mathbf{P} + \mathbf{Q}) + \mathbf{R} = \mathbf{P} + (\mathbf{Q} + \mathbf{R}),$$

by means of the construction shown in Fig. 4, where  $\vec{AB}$ ,  $\vec{BC}$ ,  $\vec{CD}$  represent  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ , respectively, and  $\vec{AD}$  either of the above sums.

Thus a unique meaning can be attached to  $\mathbf{P} + \mathbf{Q} + \mathbf{R}$ . We say that vector addition is *associative*.

**Exercise.** If

$$\mathbf{P} + 2\mathbf{Q} = \mathbf{R},$$

$$\mathbf{P} - 3\mathbf{Q} = 2\mathbf{R},$$

show that  $\mathbf{P}$  has the same direction as  $\mathbf{R}$ , and  $\mathbf{Q}$  the opposite direction.

## Components of a vector.

Let  $L$  be a straight line and  $\mathbf{P}$  a vector represented in Fig. 5 by  $\vec{AB}$ . If we draw through  $A$  and  $B$  planes perpendicular to  $L$ , these planes cut off on  $L$  a directed segment  $\vec{CD}$ , the orthogonal projection of  $\vec{AB}$ ;  $\vec{CD}$  is a common perpendicular to the pair of planes. If we take a different directed segment  $\vec{A'B'}$  to represent  $\mathbf{P}$ , we get a projection  $\vec{C'D'}$  on  $L$ . Now if we give  $\vec{AB}$  and the

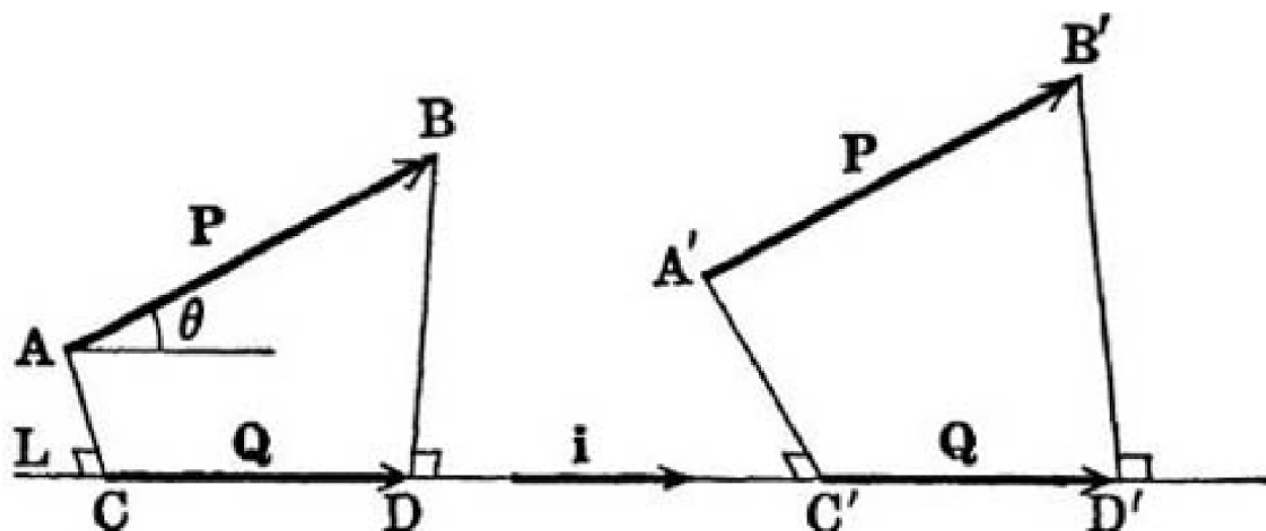


FIG. 5.—The component of a vector on a line.

pair of planes associated with it a displacement (without rotation) which carries  $A$  to  $A'$ , then  $B$  will go to  $B'$  and the pair of planes associated with

$\vec{AB}$  will coincide with the pair of planes associated with  $\vec{A'B'}$ .  $\vec{CD}$  will become a common perpendicular to the latter pair of planes, and so  $\vec{CD}$  and  $\vec{C'D'}$  are equal in magnitude and direction. Thus the vector obtained by projecting on  $L$  a directed segment representing  $\mathbf{P}$  is independent of the particular segment chosen. Writing  $\mathbf{Q}$  for the vector represented by  $\vec{CD}$  or  $\vec{C'D'}$ , we say that  $\mathbf{Q}$  is the *vector component* of  $\mathbf{P}$  on  $L$ .

Of the two senses on the line  $L$ , let us pick out one and call it the *positive* sense, the other being *negative*; the line  $L$  is then said to be *directed*. Let  $\mathbf{i}$  be a vector of unit magnitude, lying on  $L$  and pointing in the positive sense. Then it is possible to express the projection  $\mathbf{Q}$  in the form

$$\mathbf{Q} = c\mathbf{i},$$

where  $c$  is a scalar, positive if  $\mathbf{Q}$  has the same sense as  $\mathbf{i}$  and negative if  $\mathbf{Q}$  has the opposite sense to  $\mathbf{i}$ . The scalar  $c$  is called the *scalar component* of  $\mathbf{P}$  on the directed line  $L$ .

Since  $\mathbf{Q}$  is represented by any common perpendicular (in the proper sense) to the projecting planes, it is easy to see that the scalar component of  $\mathbf{P}$  on  $L$  is  $P \cos \theta$  where  $\theta$  is the angle between  $\mathbf{P}$  and the positive sense of  $L$ .

In speaking of a component (without qualification), it will be clear from the context whether the vector or scalar component is to be understood.

## Unit coordinate vectors.

In a frame of reference  $S$ , let  $Oxyz$  be rectangular Cartesian axes. Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  be unit vectors lying on  $Ox$ ,  $Oy$ ,  $Oz$ , respectively, each in the positive sense. The set  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  is called a *unit orthogonal triad*. Let  $\mathbf{P}$  be any vector, and  $P_1$ ,  $P_2$ ,  $P_3$  its components on  $Ox$ ,  $Oy$ ,  $Oz$ , respectively. These components, obtained by projection, are independent of the particular directed segment chosen to represent  $\mathbf{P}$ ; we may therefore take the representative segment with its origin at  $O$  (Fig. 6). From the rule of vector addition, it is clear that

$$(1.303) \quad \mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k}.$$



$$\mathbf{P} = \mathbf{P}(u).$$

We need no longer think of  $\mathbf{P}$  as a position vector or of  $u$  as the time.

We are familiar with the idea of differentiating a scalar function of a scalar: can we enlarge the familiar method to obtain a process for differentiating a vector function of a scalar?

We may follow the familiar plan almost word for word. We consider two values of the parameter,  $u$  and  $u + \Delta u$ , and the corresponding increment in  $\mathbf{P}$ ,

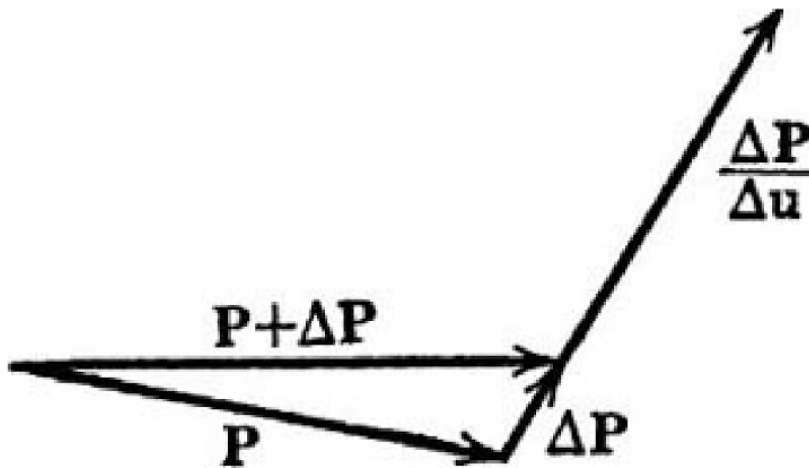


FIG. 7.—Differentiation of a vector.

$$\Delta\mathbf{P} = \mathbf{P}(u + \Delta u) - \mathbf{P}(u).$$

We multiply by  $1/\Delta u$ , to form the quotient  $\Delta\mathbf{P}/\Delta u$  (Fig. 7) and let  $\Delta u$  tend to zero. Thus we get a limiting vector

$$(1.308) \quad \frac{d\mathbf{P}}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta\mathbf{P}}{\Delta u},$$

which we call the *derivative* of  $\mathbf{P}$  with respect to  $u$ .

To find the components of the derivative we introduce unit coordinate vectors, so that

$$(1.309) \quad \mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k};$$

$P_1, P_2, P_3$  are scalar functions of  $u$ . Increasing  $u$  to  $u + \Delta u$ , we have

$$\mathbf{P} + \Delta\mathbf{P} = (P_1 + \Delta P_1)\mathbf{i} + (P_2 + \Delta P_2)\mathbf{j} + (P_3 + \Delta P_3)\mathbf{k}.$$

Subtraction gives

$$\Delta \mathbf{P} = \Delta P_1 \mathbf{i} + \Delta P_2 \mathbf{j} + \Delta P_3 \mathbf{k}.$$

Dividing by  $\Delta u$ , we get

$$\frac{\Delta \mathbf{P}}{\Delta u} = \frac{\Delta P_1}{\Delta u} \mathbf{i} + \frac{\Delta P_2}{\Delta u} \mathbf{j} + \frac{\Delta P_3}{\Delta u} \mathbf{k},$$

and so, letting  $\Delta u$  tend to zero,

$$(1.310) \quad \frac{d\mathbf{P}}{du} = \frac{dP_1}{du} \mathbf{i} + \frac{dP_2}{du} \mathbf{j} + \frac{dP_3}{du} \mathbf{k},$$

the derivatives on the right being of course derivatives of scalars—the usual derivatives of the differential calculus. Thus the components of  $d\mathbf{P}du$  are

$$\frac{dP_1}{du}, \quad \frac{dP_2}{du}, \quad \frac{dP_3}{du}.$$

In words, the components of the derivative are equal to the derivatives of the components.

The following results are easy to prove, either directly from the definition of the derivative or from (1.310):

$$(1.311) \quad \begin{cases} \frac{d}{du} (\mathbf{P} + \mathbf{Q}) = \frac{d\mathbf{P}}{du} + \frac{d\mathbf{Q}}{du}, \\ \frac{d}{du} (p\mathbf{P}) = \frac{dp}{du} \mathbf{P} + p \frac{d\mathbf{P}}{du}, \end{cases}$$

where  $\mathbf{P}, \mathbf{Q}$  are vector functions of  $u$ , and  $p$  is a scalar function of  $u$ .

It will be observed that (1.310) can be obtained directly from (1.309) by differentiation, using the rules (1.311); in this process the vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are treated as constants.

To avoid possible confusion, let us ask the question: What do we mean by saying that a vector  $\mathbf{Q}$  is constant? This is meaningless without a statement (explicit or understood) regarding the frame of reference employed. In a frame of reference  $S$ , a vector  $\mathbf{Q}$  is constant if it may be

(1.316)

$$\mathbf{f} = \dot{u}\mathbf{i} + \dot{v}\mathbf{j} + \dot{w}\mathbf{k}.$$

We could, of course, continue this process, defining a super-acceleration  $d\mathbf{f}/dt$ . However, acceleration is the important vector in Newtonian mechanics, and so we stop our definitions here.

As a simple illustration of these ideas, consider a car traveling along a straight road. It follows from (1.312) and the definition of the derivative of a vector that the velocity  $\mathbf{q}$  lies along the road and points in the direction in which the car is going. Similarly, it follows from (1.314) that the acceleration  $\mathbf{f}$  lies along the road, but now there is an important difference. The vector of acceleration does not necessarily point in the direction of motion; this is the case only if the speed is increasing. If the speed is decreasing under application of the brakes, the vector  $\mathbf{f}$  points backward. It is not hard to show that when the car rounds a curve the velocity continues to point along the road, but the acceleration points off the road toward the inside of the curve.

The formulas (1.313) and (1.315) are particularly useful for direct calculation when the motion is described by giving  $x, y, z$  as functions of  $t$ . Suppose, for example, that

$$x = a \cos \omega t, \quad y = a \sin \omega t, \quad z = 0,$$

where  $a$  and  $\omega$  are constants. Then

$$\begin{aligned}\mathbf{r} &= a \cos \omega t \cdot \mathbf{i} + a \sin \omega t \cdot \mathbf{j}, \\ \mathbf{q} &= -a\omega \sin \omega t \cdot \mathbf{i} + a\omega \cos \omega t \cdot \mathbf{j}, \\ \mathbf{f} &= -a\omega^2 \cos \omega t \cdot \mathbf{i} - a\omega^2 \sin \omega t \cdot \mathbf{j}.\end{aligned}$$

It is easy to see that this is motion in a circle, the velocity pointing along the tangent and the acceleration in along the radius.

## Units of velocity and acceleration.

In the c.g.s. system of units, the components of  $\mathbf{r}$  are measured in centimeters. Any component of the velocity  $\mathbf{q}$  is obtained by dividing a number of centimeters by a number of seconds, and the result is expressed as so many centimeters per second or, briefly,  $\text{cm. sec.}^{-1}$ . *The unit of velocity is one centimeter per second.*

Any component of the acceleration  $\mathbf{f}$  is obtained by dividing a velocity component by time or, more precisely, a number of centimeters per second by a number of seconds, and the result is expressed as so many centimeters per second per second or, briefly,  $\text{cm. sec.}^{-2}$  *The unit of acceleration is one centimeter per second per second.*

If f.p.s. units are used, the above italicized statements are modified by changing the word “centimeter” to “foot.”

Although the c.g.s. and f.p.s. units are commonly used in scientific work, there is no necessity to limit ourselves to them. The units of length and time may be chosen arbitrarily. Indeed, velocities are frequently expressed in miles per hour (m.p.h.).

Since velocity is obtained by dividing length by time, we say it has the “dimensions”  $[LT^{-1}]$ ; similarly, acceleration has the dimensions  $[LT^{-2}]$ . This notation is discussed in the Appendix.

## Gradient vector.

When to each point of space, or to each point of a plane, a scalar is assigned, we say that we are dealing with a *scalar field*. Thus the distribution of pressure (or temperature) in the atmosphere at a certain time gives a scalar field. Or consider a map on which heights are marked; the height gives a scalar field over the map.

When to each point of space or to each point of a plane a vector is assigned, we have a *vector field*. The wind velocity in the atmosphere gives a vector field.

We shall now show that a scalar field defines an associated vector field in a very simple way. For generality, we shall make the argument three-dimensional, but the reader will find it interesting to consider by way of illustration the map marked with heights.

Let  $Oxyz$  be rectangular Cartesian axes and  $V(x, y, z)$  a scalar field. (For the map, we suppress  $z$ ;  $V(x, y)$  is the height of the land at the point  $x, y$ .) The surfaces  $V = \text{constant}$  are called *level surfaces*. (On the map the level surfaces become the contour lines.) Let  $A$  be any point and  $S$  the level surface passing through  $A$  (Fig. 8). Let us draw the normal to  $S$  on the side on which  $V$  increases, and proceed an arbitrary distance  $n$  along this normal. Then  $V$  is a function of  $n$ , and

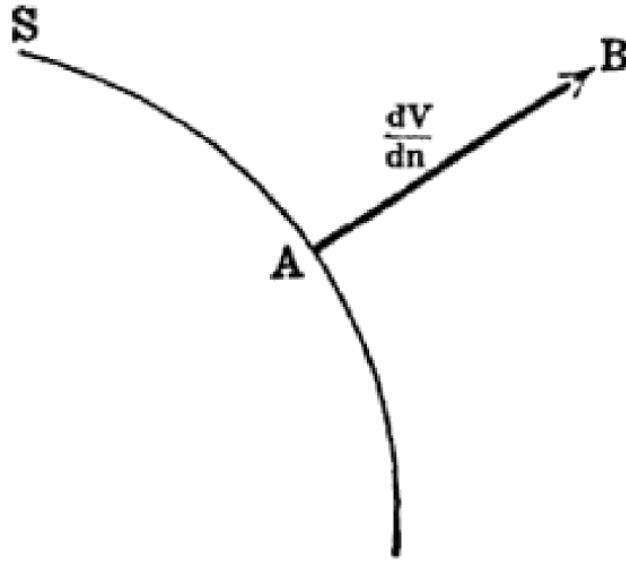


FIG. 8.—The gradient vector.

$$\frac{dV}{dn} > 0$$

at  $A$ , since  $V$  is increasing.

We now introduce a vector  $\vec{AB}$ , called the *gradient* of  $V$  at  $A$ , or briefly *grad*  $V$ . (It is also denoted by  $\nabla V$ .) The defining properties of *grad*  $V$  are as follows:

(i) Its direction is along the normal to  $S$  at  $A$ , in the sense of  $V$  increasing.

(ii) Its magnitude is  $dV/dn$ , calculated at  $A$ .

In the case of the map, *grad*  $V$  is perpendicular to the contour line and points in the uphill sense; its magnitude is the rate of increase of height.

We now proceed to find the components of *grad*  $V$  on the axes of coordinates. Let  $\alpha, \beta, \gamma$  be the direction cosines of the normal to  $S$  in the sense of  $V$  increasing. Then any infinitesimal displacement lying in  $S$ , i.e., making

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 0,$$

also makes  $\alpha dx + \beta dy + \gamma dz = 0$ . Hence,

$$(1.317) \quad \frac{\partial V}{\partial x} = \phi\alpha, \quad \frac{\partial V}{\partial y} = \phi\beta, \quad \frac{\partial V}{\partial z} = \phi\gamma,$$

$$\frac{\partial V}{\partial x} \frac{dx}{ds} + \frac{\partial V}{\partial y} \frac{dy}{ds} + \frac{\partial V}{\partial z} \frac{dz}{ds} = \frac{dV}{ds}.$$

*The component of grad  $V$  in any direction is the rate of change of  $V$  in that direction.*

We have seen how to obtain a vector field (grad  $V$ ) from a scalar field ( $V$ ). It is not in general possible to express an *arbitrary* vector field as the gradient of a scalar field, but we find in mathematical physics many vector fields that can be so expressed. In Sec. 2.4 we shall discuss fields of force; in most cases of physical interest a field of force is the gradient of a scalar field of potential energy (with a change of sign). Since the description of a vector field requires three functions and a scalar field only one function, a considerable simplification results from the use of the scalar field.

*Exercise.* If in a plane  $V = x^2 + y^2$ , find the component of grad  $V$  at the point (1, 0) in a direction making an angle of  $45^\circ$  with the  $x$ -axis.

## 1.4. FUNDAMENTAL LAWS OF NEWTONIAN MECHANICS

Let us suppose that we are conducting experiments in which small bodies of measured masses are acted on by measured forces. We choose some frame of reference and observe the motions of the bodies relative to it. We shall use the following notation:

$\mathbf{P}$  = force,

$m$  = mass,

$\mathbf{f}$  = acceleration.

The units of force, mass, length, and time are chosen arbitrarily.

We ask this question: As a physical fact, is there any simple relation connecting  $\mathbf{P}$ ,  $m$ , and  $\mathbf{f}$  for the motion of a body? The answer to this question is, in general: There is no simple relation.

We have thought of the experiments as conducted in any frame of reference—it might be the cabin of an airplane looping the loop, or it might be an ordinary laboratory. We ask a second question: Is it possible to *choose* a frame of reference so that there is a simple relation connecting  $\mathbf{P}$ ,  $m$ , and  $\mathbf{f}$ ? The answer is: Yes.

One frame of reference yielding a simple relation among  $\mathbf{P}$ ,  $m$ , and  $\mathbf{f}$



LAW OF THE PARALLELOGRAM OF FORCES. *When two forces  $\mathbf{P}$  and  $\mathbf{Q}$  act on a particle, they are together equivalent to a single force  $\mathbf{P} + \mathbf{Q}$ , the vector sum being defined by the parallelogram construction as in Sec. 1.3.*

It is usual to call  $\mathbf{P} + \mathbf{Q}$  the *resultant* of  $\mathbf{P}$  and  $\mathbf{Q}$ ;  $\mathbf{P}$  and  $\mathbf{Q}$  are called the *vector components* of  $\mathbf{P} + \mathbf{Q}$ .<sup>†</sup>

In setting up a system of laws or axioms, it is generally considered desirable to make them independent, in the sense that no one of them can be deduced from the others. Many attempts have been made to deduce the parallelogram of forces, but all these deductions require the statement of other laws, which are individually simpler than the parallelogram law but rather long to state; so, for brevity, we accept the parallelogram law directly.\*

The three laws stated above form the logical basis of mechanics. Their full meaning can be understood only by applying them, and we shall not delay the development of the subject by further general discussion. We should, however, point out their significance for the prediction of the results of ordinary laboratory experiments.

Let us suppose that we are discussing the motion of a billiard ball which rolls down an inclined plane in a laboratory. Our problem is to predict the behavior of the ball by mathematical reasoning. On the one hand, we have the actual physical apparatus, on the other our mathematical model, in which the ball, the plane, and the forces acting are replaced by their mathematical idealizations. There is a precise *correspondence* between the physical things and the ingredients of our mathematical model.

But here there enters an important question: What physical frame of reference corresponds to the basic Newtonian frame mentioned in the law of motion—the frame relative to which (1.401) holds? We have already indicated the answer: The physical frame in question is the astronomical frame of reference.

But we do not want to know the behavior of the billiard ball relative to the astronomical frame of reference; we want to know its behavior relative to the walls and floor of the laboratory, i.e., relative to the earth's surface. Is it legitimate to regard the earth's surface as a physical frame corresponding to the Newtonian frame of (1.401)? Strictly speaking, it is

unit of force is used, so that the law of motion is (1.402). In statical problems, on the other hand, we shall leave the unit of force arbitrary, since there is no additional simplicity to be gained by restricting it.

At this point the reader is advised to study the Appendix at the end of the book in order that he may understand the references in the text to units and dimensions. In particular, attention is directed to the use of the theory of dimensions as an easy and rapid check against slips in calculation—a check which is of great value both in elementary and advanced work. *In any equation in mechanics, all terms must have the same dimensions.* For example, if we had carelessly derived the formula (cf. page 28)

$$\mathbf{f} = \alpha\omega \cos \omega t \cdot \mathbf{i} - \alpha\omega^3 \sin \omega t \cdot \mathbf{j},$$

a glance would show that something is wrong with this formula. The acceleration  $\mathbf{f}$  has dimensions  $[LT^{-2}]$ ,  $\alpha\omega$  has dimensions  $[LT^{-1}]$ ,  $\alpha\omega^3$  has dimensions  $[LT^{-3}]$ , while the trigonometrical functions and the vectors  $\mathbf{i}$ ,  $\mathbf{j}$  are dimensionless.

## 1.5. SUMMARY OF THE FOUNDATIONS OF MECHANICS

The purposes of this first chapter have been (i) to dig down to the fundamental physical ideas and (ii) to lay the foundations of a logical structure. Before proceeding to the next chapter, we now extract and emphasize those concepts and laws which will be required later.

### I. The ingredients of mechanics.

(a) A *particle* has position and mass ( $m$ ).

(b) A *rigid body* is a system of particles, the distances between which remain unchanged. It may also be regarded as a continuous distribution of matter.

(c) A *frame of reference* is a rigid body in which axes of coordinates are taken.

(d) A *force* has point of application, direction, and magnitude.

(e) The *units* of mass, length, and time are arbitrary. So also is the unit of force, but it is convenient in dynamics to connect the unit of force with the units of mass, length, and time.

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

**2.** A particle is acted on by forces of magnitudes  $P$  and  $Q$ , their lines of action making an angle  $\theta$  with one another. They are to be balanced by two equal forces, acting at right angles to one another. Find the common magnitude of these forces.

**3.** Show that if the magnitudes of a number of coplanar vectors are multiplied by a common factor, the directions of the vectors being unchanged, the magnitude of the sum is multiplied by the same factor and its direction is unchanged. Show also that if all the vectors are rotated through a common angle in their plane, without change of magnitude, their sum is rotated through the same angle without change of magnitude.

**4.** Forces of magnitudes 3, 4, and 5 lb. wt. act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant.

**5.** Two forces acting in opposite directions on a particle have a resultant of 34 lb. wt.; if they acted at right angles to one another, their resultant would have a magnitude of 50 lb. wt. Find the magnitudes of the forces.

**6.** An airplane dives at 400 miles per hour, losing height at the rate of 220 ft. per sec. What is the horizontal component of its velocity in miles per hour?

**7.** Coplanar forces of magnitudes  $P$ ,  $2P$ ,  $4P$  act on a particle. How should they be directed to make the resultant (i) a maximum, (ii) a minimum?

**8.** If any number of coplanar vectors all of the same magnitude are drawn from a point, arranged symmetrically so that the angles between adjacent vectors are all equal, prove that their sum is zero.

**9.** If  $V = x^2 + y^2 + z^2 + xy + x$ , at what points in space is the vector  $\text{grad } V$  parallel to the  $z$ -axis?

**10.** A scalar field is given over a plane by

$$V = \frac{x^2 + y^2}{2x}.$$

What are the level curves? Show that, at the point with polar

where  $a$  and  $p$  are constants. Express the force acting on it as a function of  $x$ .

**18.** A tug tows a barge  $A$ , which in turn tows another barge  $B$ . They start to move with an acceleration  $f$ . Find the tensions in the towing cables, in terms of  $f$  and  $m, m'$  (the masses of the barges).

**19.** Indicate the fallacy in the following argument: A locomotive pulls a train. But to every action there is an equal and opposite reaction. Therefore the train pulls the locomotive backward with a force equal to the pull of the locomotive, and so there can be no motion.

**20.** If the fundamental law of mechanics for a particle moving on a straight line were

$$m \frac{d}{dt} \left( \frac{\dot{x}}{\sqrt{1 - (\dot{x}^2/c^2)}} \right) = P,$$

instead of (1.402),  $m$  and  $c$  being constants, find the distance traveled from rest in time  $t$  under the action of a constant force  $P$ . (This is the relativistic equation of motion; see [Chap. XVI.](#))

\* The reader will of course understand that when we speak of a “model” we do not mean an actual physical reproduction on a small scale. We use the word—for want of a better—to describe our simplified mental picture of a physical object.

\* See treatments in E. T. Whittaker, *Analytical Dynamics* (Cambridge University Press, 1927), p. 29; H. Lamb, *Statics* (Cambridge University Press, 1928), p. 12, and *Dynamics* (Cambridge University Press, 1929), p. 17; J. S. Ames and F. D. Murnaghan, *Theoretical Mechanics* (Ginn and Company, Boston, 1929), p. 104. For a critical and historical account of the development of mechanics, see E. Mach, *The Science of Mechanics* (Open Court Publishing Company, Chicago, 1919).

\* The word “vector” is derived from the Latin *veho*, I carry.

\* Thus velocity is a vector and speed is a scalar. However, when no confusion is likely to arise, the word “velocity” is often used in the scalar sense to denote the magnitude of the velocity vector; for example, the expression “velocity of light” is used instead of the correct expression “speed of light.”

\* More accurately, the mass center of the solar system (see Sec. 3.1); actually this point is not far from the center of the sun.

† This relation is in excellent agreement with astronomical observations, but there are exceptions; the orbit of the planet Mercury reveals a minute discrepancy. Although few modern astronomers accept this relation as absolute physical truth, its validity is so high that it is taken as the basis of celestial mechanics. To take a deeper point of view, we must recast our whole mode of thought and follow the general theory of relativity.

(i) The statics and dynamics of a system of particles lying in a fixed plane.

(ii) The statics and dynamics of rigid bodies which can move only parallel to a fixed plane, the displacement or velocity of every particle being parallel to the fixed plane.

As an example of (i), we may mention the problem of the motion of the earth relative to the sun, both being treated as particles, and as an example of (ii) the motion of a cylinder rolling down an inclined plane.

The plane in which the system lies, or to which the motion is parallel, will be called the *fundamental plane*.

We shall find in both statics and dynamics that the basic laws lead to certain general principles or methods. When any specific problem presents itself, we do not attack it directly from first principles as a rule; we can avoid waste of energy by applying to the problem one of the general methods. The present chapter is devoted to general methods in plane statics, with inclusion of the space theory where it presents no difficulty.

## 2.2. EQUILIBRIUM OF A PARTICLE

According to (1.401) a particle will have an acceleration unless the force acting on it vanishes. Thus if  $\mathbf{P}$  is the force acting on a particle, the necessary and sufficient condition for equilibrium is

$$(2.201) \quad \mathbf{P} = \mathbf{0}.$$

If several forces  $\mathbf{P}, \mathbf{Q}, \mathbf{R} \dots$  act on a particle, the necessary and sufficient condition for equilibrium is the vanishing of the resultant force, i.e.,

$$(2.202) \quad \mathbf{P} + \mathbf{Q} + \mathbf{R} + \dots = \mathbf{0}.$$

This vector condition may also be expressed in scalar form by means of components. Let a particle be acted on by forces whose components on an orthogonal triad are indicated as follows:

$$\begin{aligned} &\mathbf{P}(P_1, P_2, P_3), \\ &\mathbf{Q}(Q_1, Q_2, Q_3), \\ &\mathbf{R}(R_1, R_2, R_3), \\ &\dots\dots\dots \end{aligned}$$

Then the conditions for equilibrium are



$$(2.203) \quad \begin{cases} P_1 + Q_1 + R_1 + \dots = 0, \\ P_2 + Q_2 + R_2 + \dots = 0, \\ P_3 + Q_3 + R_3 + \dots = 0. \end{cases}$$

All the above remarks hold whether the forces acting on the particle lie in a plane or not. The particular feature of the plane case is that only two components are to be considered instead of three.

No difficulty will be found in using (2.202) to prove the following theorems which are often useful:

(i) **THE TRIANGLE OF FORCES.** If a particle is in equilibrium under the action of three forces, these forces may be represented in magnitude and direction by the three sides of a triangle, taken in order (Fig. 9).

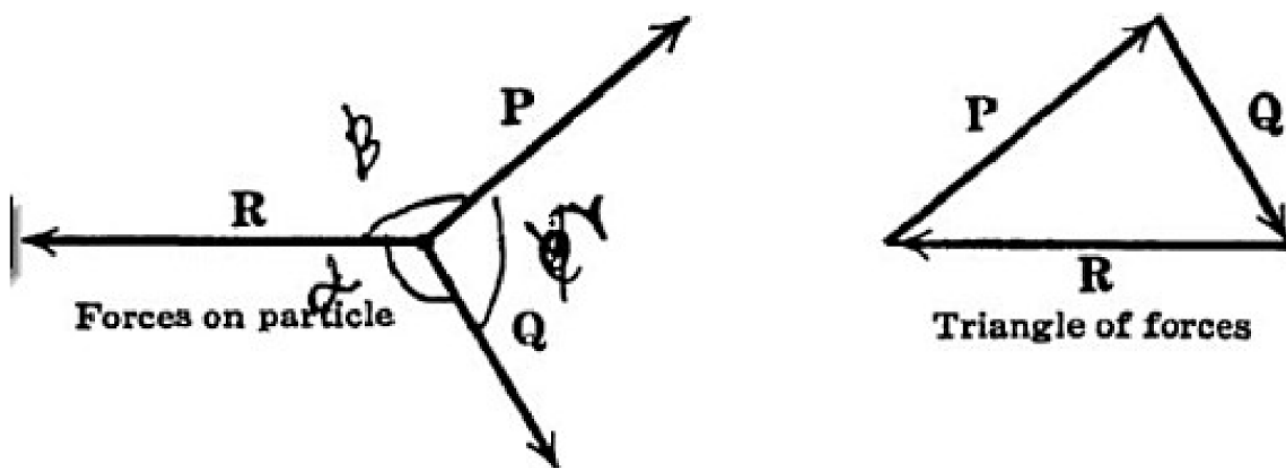


FIG. 9.—The triangle of forces.

(ii) **THE POLYGON OF FORCES.** If a particle is in equilibrium under the action of several forces, these forces may be represented by the sides of a closed polygon, taken in order.

(iii) **LAMY'S THEOREM.** If a particle is in equilibrium under the action of three forces **P**, **Q**, **R**, then

$$(2.204) \quad \frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma},$$

where  $\alpha$  is the angle between **Q** and **R**,  $\beta$  the angle between **R** and **P**, and  $\gamma$  the angle between **P** and **Q**.

*Exercise.* A particle is in equilibrium under three forces. Two of the forces act at right angles to one another, one being double the other. The third force has a magnitude 10 lb. wt. Find the magnitudes of the other two.

## 2.3. EQUILIBRIUM OF A SYSTEM OF PARTICLES



## Systems of particles.

Let us now consider a system of particles. We must in all cases come to a clear understanding as to what is included in the system under consideration. Let us suppose that we are dealing with a book which rests on a table, the table standing on the floor. The book and the table are regarded (in our mathematical model) as composed of a very great number of particles. In talking about this arrangement of matter, we are not compelled to think of the “system” under consideration as composed of the book and the table. If we like, we may think of the table alone as a system, or the book alone as a system, or the hundredth page of the book as a system.

It is important to realize that the system under consideration is something we pick out from the given arrangement in an arbitrary manner. It is necessary to understand this in order to appreciate the distinction between *external* and *internal* forces.

## External and internal forces.

The book presses down on the table and the table presses up on the book with an equal and opposite force (law of action and reaction). If the system is book + table, these forces are both exerted by particles of the system. But if the system consists of the book only, this is no longer the case; the force exerted by the table on the book is due, not to particles of the system, but to particles lying outside the system.

We make the following general definition:

A force acting on a particle of a given system is an *internal* force when it is exerted by another particle of that system; otherwise it is an *external* force.

In accordance with the law of action and reaction, internal forces occur in equal and opposite pairs, each pair representing the mutual interactions of a pair of particles of the system.

Figure 10 shows three particles  $A$ ,  $B$ ,  $C$ , the broken line indicating the boundary of the system. The forces are classified as follows:

External:  $\mathbf{P}$  at  $A$ ,  $\mathbf{Q}$  at  $B$ ,  $\mathbf{R}$  at  $C$ .

Internal:  $\mathbf{U}$  at  $B$ ,  $-\mathbf{U}$  at  $C$ ,  $\mathbf{V}$  at  $C$ ,  $-\mathbf{V}$  at  $A$ ,  $\mathbf{W}$  at  $A$ ,  $-\mathbf{W}$  at  $B$ .

Figure 11 shows these particles situated precisely as in Fig. 10 and subject to the same forces. The only difference between the two diagrams lies in the position of the broken line, indicating the boundary of the system under consideration. In Fig. 11 the system contains only the particles  $A$  and  $B$ . Now  $\mathbf{P}$  at  $A$ ,  $-\mathbf{V}$  at  $A$ ,  $\mathbf{Q}$  at  $B$ ,  $\mathbf{U}$  at  $B$  are external; and  $\mathbf{W}$  at  $A$ ,  $-\mathbf{W}$  at  $B$  are internal. If

we reduce the system under consideration to one particle only, there are no longer any internal forces.

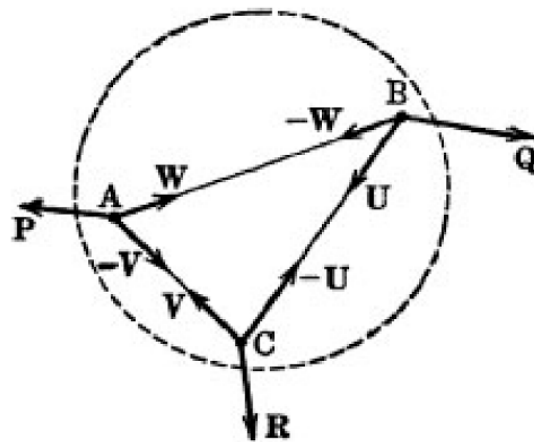


FIG. 10.—External and internal forces. The “system” is enclosed by the broken line.

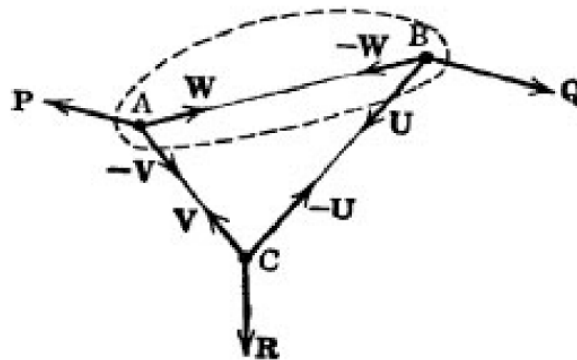


FIG. 11.—The same particles and forces as in [Fig. 10](#), but a different “system.”

The only essential difference between the treatment of this question of internal and external forces in a plane and its treatment in space is that in the former case we may delimit the system by a closed curve, whereas in the latter case we require a closed surface.

### Necessary conditions for equilibrium (forces).

We shall now obtain necessary conditions for the equilibrium of any system of particles. The meaning of the word “necessary” should be emphasized: these conditions must be satisfied if the system is in equilibrium, but the satisfaction of the conditions does not imply that there is equilibrium.

Consider any particle of the system, assumed in equilibrium. That particle is itself in equilibrium, and hence the vector sum of all forces acting on it is zero. Similarly for all particles. Hence,

*The vector sum of all forces acting on all particles is zero.*

But the forces are some external, some internal. We may state

*The vector sum of all external forces and all internal forces is zero.* From

indicating rotations in one sense, we use the  $-$  sign for vectors indicating rotations in the opposite sense. The significance of the signs will be better understood when Sec. 9.3 has been read, but for the present the following description of the convention will serve.

Let us suppose the line  $L$  drawn perpendicular to the plane of the paper, intersecting it at the point  $A$  (Figs. 13a and b); the moment is  $+aP$  when  $\mathbf{P}$  indicates a counterclockwise rotation around  $A$  in the plane, and  $-aP$  when a clockwise rotation is indicated.

Consider now a line  $L$  and a bound vector  $\mathbf{P}$  which is not perpendicular to  $L$ . The moment of  $\mathbf{P}$  about  $L$  is defined to be the moment about  $L$  of the projection of  $\mathbf{P}$  on a plane perpendicular to  $L$ , the latter moment having already been defined above.

The following facts are now obvious:

(i) The moment of  $\mathbf{P}$  about  $L$  is unaltered if  $\mathbf{P}$  is made to slide along its line of action without change of magnitude or sense. (It is seen at once that this only slides the projection along its line of action, without change of magnitude or sense.)

(ii) The sum of the moments about any line  $L$  of two vectors  $\mathbf{P}$ ,  $-\mathbf{P}$  situated on the same line is zero. (Their moments are equal in magnitude but opposite in sign.)

(iii) The moment about  $L$  of a vector  $\mathbf{P}$  is unaltered if  $\mathbf{P}$  is moved without change of magnitude or direction in a direction parallel to  $L$ . (This does not change its projection on a plane perpendicular to  $L$ .)

When we deal with the moments of coplanar vectors about a line perpendicular to their plane, we often refer to these moments as moments about a point, namely, the point where the line cuts the plane.

$\mathbf{R} + \dots$  with origin  $B$ .

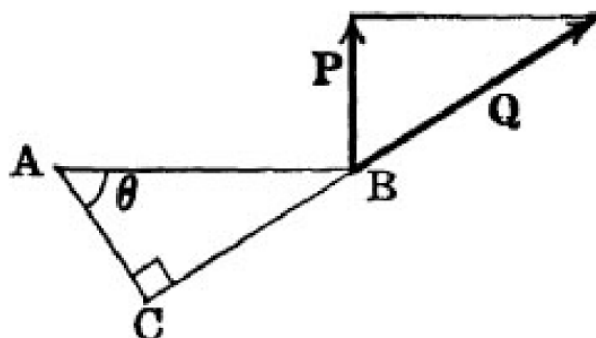


FIG. 15.—The plane containing  $\mathbf{P}$  and  $\mathbf{Q}$ .

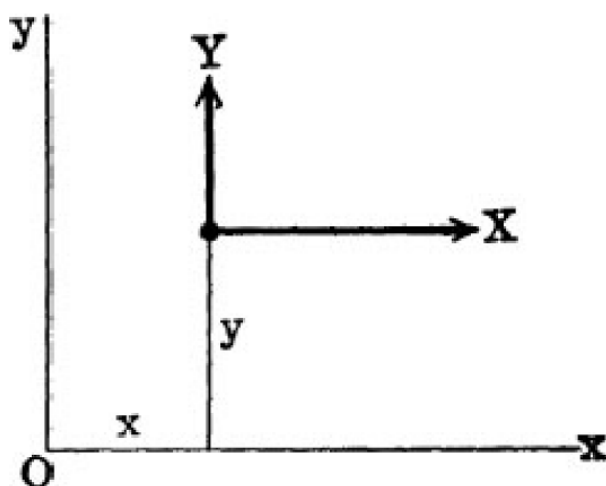


FIG. 16.—Analytical method of finding the moment of a vector.

In statics the only vectors whose moments we have occasion to consider are forces. But it should be noted that the above definitions and theorems hold for any vectors. We shall have occasion to use them in [Chap. V](#) in discussing angular momentum.

The following analytical result is important; it is obvious from [Fig. 16](#), which shows the projection on the plane  $Oxy$ . *If a vector with components  $(X, Y, Z)$  acts at the point  $(x, y, z)$ , then its moment about the perpendicular to the plane  $Oxy$  at  $O$  is*

$$(2.303) \quad M = xY - yX.$$

The moment about the perpendicular to the plane  $Oxy$  at the point  $(a, b)$  is

$$(2.304) \quad M = (x - a)Y - (y - b)X.$$

These formulas take care of the sign of  $M$  automatically.

*Exercise.* A force, of fixed magnitude  $R$  and variable inclination  $\theta$  to the  $x$ -axis, acts in the plane  $Oxy$  at the fixed point  $(a, b)$ . Find its moment about the origin as a function of  $\theta$ , and obtain the values of  $\theta$  for which this moment (i) is a maximum, (ii) is a minimum, (iii)

vanishes.

### **Necessary conditions for equilibrium (moments).**

Let us now return to the consideration of a system of particles in equilibrium. Let  $L$  be any line. Consider any particle. Since it is in equilibrium, the resultant of all the forces acting on it is zero; hence, by Varignon's theorem the sum of the moments about  $L$  of all forces acting on the particle is zero. Similarly for all particles. Hence,

*The sum of the moments about  $L$  of all external forces and all internal forces is zero.*

But the sum of the moments of a pair of equal and opposite internal forces is zero. Hence,

*The sum of the moments about  $L$  of all internal forces is zero.* Comparing this with the preceding statement, we have

(2.305) *The sum of the moments about  $L$  of all external forces is zero.*

From (2.301) and (2.305) we may now state necessary conditions for the equilibrium of any system of particles.

*If a system of particles is in equilibrium, then*

(i) *The vector sum of all EXTERNAL forces is zero.*

(ii) *The sum of the moments of all EXTERNAL forces about any line is zero.*

This result is the key to the solution of statical problems, and it should be remembered.

The particular form of the above statement applicable to plane statics is as follows:

*If a system of particles is in equilibrium, then*

$$(2.306) \quad \mathbf{F} = \mathbf{0}, \quad N = 0,$$

*where  $\mathbf{F}$  is the vector sum of the projections of all EXTERNAL forces on the fundamental plane, and  $N$  the sum of the moments of all EXTERNAL forces about any line perpendicular to the fundamental plane.*

It is convenient to have an explicit form of (2.306). Let axes be chosen so that the fundamental plane is  $z = 0$ . We shall take moments about the  $z$ -axis. Suppose now that the system is acted on by external forces with components  $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2), \dots, (X_n, Y_n, Z_n)$  at points  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$ . By projection on the fundamental plane, it follows that  $\mathbf{F}$  and  $N$  are unaltered if we replace this system by forces in the fundamental plane with components  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  at points  $(x_1, y_1), (x_2, y_2), \dots, (x_n,$



$y_n$ ). Hence, by (2.303) we see that if  $(X, Y)$  are the components of  $\mathbf{F}$ , then

$$(2.307) \quad X = \sum_{i=1}^n X_i, \quad Y = \sum_{i=1}^n Y_i, \quad N = \sum_{i=1}^n (x_i Y_i - y_i X_i),$$

and so necessary conditions for equilibrium are

$$(2.308) \quad \sum_{i=1}^n X_i = 0, \quad \sum_{i=1}^n Y_i = 0, \quad \sum_{i=1}^n (x_i Y_i - y_i X_i) = 0.$$

It might be thought that, by taking moments about other lines perpendicular to the fundamental plane, new conditions might be obtained; but this is not so. For, by (2.304), if moments are taken about a perpendicular to the plane at  $(a, b)$ , then the total moment is

$$\begin{aligned} \sum_{i=1}^n [(x_i - a) Y_i - (y_i - b) X_i] &= \sum_{i=1}^n (x_i Y_i - y_i X_i) \\ &\quad - a \sum_{i=1}^n Y_i + b \sum_{i=1}^n X_i, \end{aligned}$$

and this vanishes automatically if (2.308) are satisfied. Thus conditions of the type (2.306) or (2.308) are actually three in number, and no more.

The following important results are easy to establish from the principles laid down above:

(i) If a system is in equilibrium under the action of only two external forces, then these forces have a common line of action, equal magnitudes, and opposite senses.

(ii) If a system is in equilibrium under the action of only three external forces, these forces lie in a plane and their lines of action are either concurrent or parallel.

If a force is measured in dynamical units, its moment has dimensions  $[ML^2T^{-2}]$  (see Appendix). In the e.g.s. system, the unit moment is 1 dyne cm. or 1 gm. cm.<sup>2</sup> sec.<sup>-2</sup>; in the f.p.s. system, it is 1 ft. poundal or 1 lb. ft.<sup>2</sup> sec.<sup>-2</sup>. In statics we frequently use a unit of force which is not a dynamical unit, such as the lb. wt. or the ton wt. (contracted to read lb. and ton). The corresponding moments are measured in ft. lb. or ft. ton.

## Equipollent systems of forces.

Two systems of forces are said to be *equipollent*\* when the following

conditions are satisfied:

(i) The vector sum of all the forces of one system is equal to the vector sum of all the forces of the other system.

(ii) The sum of the moments of all the forces of one system about an arbitrary line is equal to the sum of the moments of all the forces of the other system about that line.

For the discussion of plane mechanics, we require only a restricted type of equipollence, which we shall call *plane equipollence*. Two systems of forces are said to be *plane-equipollent* for the fundamental plane if

(i) The vector sum of the projections on the fundamental plane of all the forces of one system is equal to the vector sum of the projections on that plane of all the forces of the other system.

(ii) The sum of the moments of all the forces of one system about an arbitrary line perpendicular to the fundamental plane is equal to the sum of the moments of all the forces of the other system about the same line.

It is evident that if  $\mathbf{F}$ ,  $\mathbf{F}'$  are the vector sums of the projections of the forces of the two systems, and  $N$ ,  $N'$  the moments about some line, then the conditions for plane equipollence are

$$(2.309) \quad \mathbf{F} = \mathbf{F}', \quad N = N'.$$

If  $\mathbf{F} = \mathbf{0}$ ,  $N = 0$ , we say that the system is *plane-equipollent to zero*. Thus, if a system of particles is in equilibrium, the forces acting on it are plane-equipollent to zero.

The idea of equipollence is extremely useful in statics. The solutions of problems in plane statics turn on the conditions (2.306), and difficulties may arise in the calculation of  $\mathbf{F}$  and  $N$ . These difficulties are reduced by splitting up the calculation into parts, each of which is simple. This reduction depends on the following fact (obvious from the definition of equipollence): *For the calculation of the vector sum of the projections of the forces of a system on the fundamental plane and the sum of their moments about a line perpendicular to that plane, any system of forces may be replaced by a system plane-equipollent to it.*

In what follows below, we shall speak only of forces in the fundamental plane. This makes for simplicity of expression without any real loss of generality, because in problems of plane mechanics we are actually interested in projections on the fundamental plane and moments about lines perpendicular to it; for the calculation of these, we may replace a given force system by a plane-equipollent system in the fundamental plane.

*Exercise.* Find a system of two forces equipollent to a system of three forces represented by the sides of an equilateral triangle taken in order.

## Couples.

A couple is defined as a pair of forces acting on parallel lines, equal in magnitude and opposite in sense. In fact, a couple consists of a pair of forces  $\mathbf{P}$ ,  $-\mathbf{P}$ . A line drawn perpendicular to the two lines of action and terminated by them is called the *arm* of the couple.

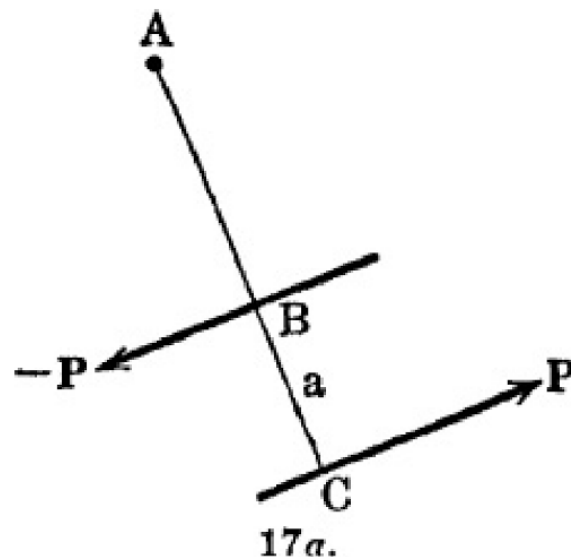
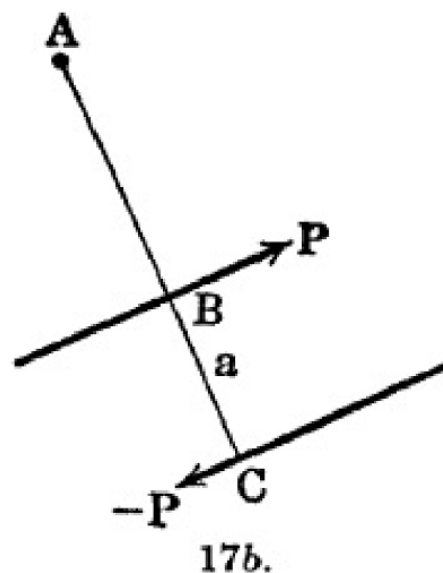


FIG. 17.—(a) Couple with positive moment.



(b) Couple with negative moment.

Thus the vector sum of the forces constituting a couple is zero. Consider now the sum of the moments of the forces forming a couple about any line  $L$ , perpendicular to the plane of the couple and cutting it at  $A$  (Figs. 17a and 17b). If  $AB$ ,  $AC$  are the perpendiculars dropped from  $A$  on the lines of action, the sum

of the moments is

$$M = P \cdot AC - P \cdot AB = P \cdot BC = Pa \quad (\text{Fig. 17a}),$$

$$M = P \cdot AB - P \cdot AC = -P \cdot BC = -Pa \quad (\text{Fig. 17b}),$$

where  $a$  is the arm of the couple. The rule for sign is easily seen to be as follows:

$$(2.310) \quad M = \pm Pa,$$

where the  $+$  or  $-$  sign is to be taken according as the forces indicate a positive (counterclockwise) rotation or a negative (clockwise) rotation in the plane of the couple about any point taken between their lines of action.

We observe that the sum of the moments of the forces forming a couple about a line perpendicular to its plane is the same for all such lines. Hence, we may speak in an absolute sense of the *moment of a couple*. Since the vector sum of forces in a couple is zero, it follows that *two couples in the fundamental plane are plane-equipollent if they have the same moment*.

It is evident that two couples of moments  $M, M'$  in the fundamental plane are together plane-equipollent to a single couple of moment  $M + M'$  in that plane.

### Reduction of a general plane force system.

Consider a system of forces in the fundamental plane. Let  $\mathbf{F}$  be their vector sum and  $N$  the sum of their moments about some point  $O$  in the plane.\* Consider on the other hand a single force  $\mathbf{F}$  at  $O$  and a couple of moment  $N$  in the fundamental plane. Obviously, this simple system is plane-equipollent to the given system. Hence, we may state the following general result:

*A system of forces in the fundamental plane is plane-equipollent to a single force applied at an arbitrary point in the plane, together with a couple.*

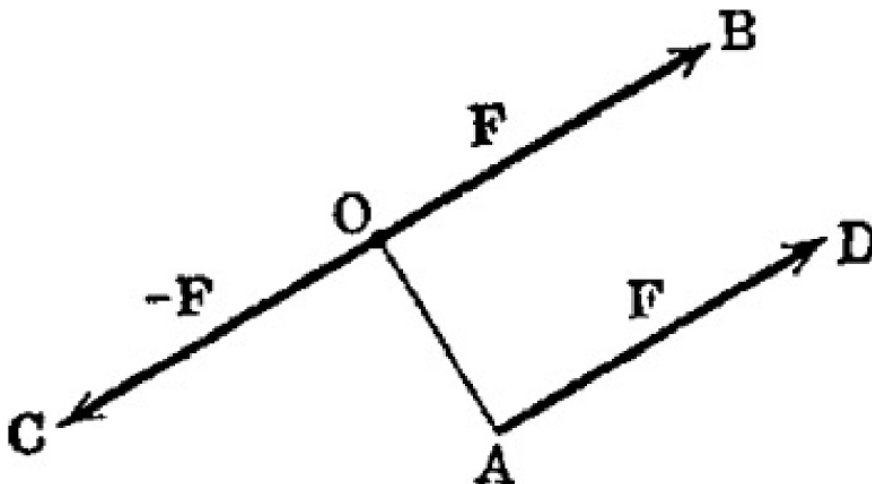


FIG. 18.—Reduction of a force and a couple to a single force.

It is customary also to express this by saying that a system of forces in the fundamental plane may be *reduced* to a force and a couple.

Expressed analytically, a system of forces  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  at points  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  may be reduced to a single force at the origin with components  $X, Y$ , together with a couple  $N$ , where

$$(2.311) \quad X = \sum_{i=1}^n X_i, \quad Y = \sum_{i=1}^n Y_i, \quad N = \sum_{i=1}^n (x_i Y_i - y_i X_i).$$

We shall now show that a still greater reduction is possible.

Let  $O$  be an arbitrary point. The force system, as we already know, may be reduced to a force  $\mathbf{F}$  at  $O$ , together with a couple of moment  $N$ . In Fig. 18, the force  $\mathbf{F}$  at  $O$  is shown as  $\vec{OB}$ . We now consider the following two cases:

CASE (i):  $\mathbf{F} \neq \mathbf{0}$ . Let us suppose  $N$  positive for simplicity, the case where  $N$  is negative being similarly dealt with. We draw  $OA$  perpendicular to  $OB$  as in Fig. 18 and measure off  $OA$  equal to  $N/F$ . Then the couple is equipollent to the pair of forces

$$\vec{OC} \text{ (or } -\mathbf{F} \text{ at } O); \quad \vec{AD} \text{ (or } \mathbf{F} \text{ at } A).$$

Thus the given system is reduced to the three forces

$$\mathbf{F} \text{ at } O, \quad -\mathbf{F} \text{ at } O, \quad \mathbf{F} \text{ at } A.$$

These are equipollent to a single force  $\mathbf{F}$  at  $A$ .

CASE (ii):  $\mathbf{F} = \mathbf{0}$ . Here the system is reduced to a couple.

Hence we may state the following general result:

*Any force system in the fundamental plane may be reduced either to a single force or to a couple.*

It may be noted that reduction to a single force will occur much more frequently than reduction to a couple, because reduction to a couple occurs only when a special condition is satisfied, viz.,  $\mathbf{F} = \mathbf{0}$ , or, in other words, when the vector sum of the forces in the system is zero.

Let us now carry out this reduction to a single force or to a couple analytically, the given plane force system being specified as consisting of forces with components  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  at points with coordinates  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ . Let us suppose that this system may be reduced to a force with components  $(X, Y)$  at  $(x, y)$ . The conditions of plane equipollence are



$$(2.312) \quad X = \sum_{i=1}^n X_i, \quad Y = \sum_{i=1}^n Y_i, \\ xY - yX = \sum_{i=1}^n (x_i Y_i - y_i X_i).$$

The first two equations determine the components of the single force. The last equation gives one relation between the coordinates of the point of application; this relation, being linear, defines a straight line. It is seen at once that this line points in the direction of the force with components  $(X, Y)$ ; it is, in fact, the line of action of that force. The last equation of (2.312) leaves the point of application indeterminate to a certain extent—it may take any position on a certain line. One particular point on this line is given by the symmetrical formulas

$$(2.313) \quad x = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n Y_i}, \quad y = \frac{\sum_{i=1}^n y_i X_i}{\sum_{i=1}^n X_i}.$$

If it should happen that

$$(2.314) \quad \sum_{i=1}^n X_i = 0, \quad \sum_{i=1}^n Y_i = 0, \quad \sum_{i=1}^n (x_i Y_i - y_i X_i) \neq 0,$$

it is evident that we cannot find  $X, Y, x, y$  to satisfy (2.312). Then the system cannot be reduced to a single force. To find the couple to which it can be reduced, we compare it with the couple consisting of forces  $(0, -Y), (0, Y)$  at the points  $(0, 0), (0, x)$ , respectively. The conditions of equipollence are

$$0 + 0 = 0, \quad -Y + Y = 0, \quad xY = \sum_{i=1}^n (x_i Y_i - y_i X_i).$$

These equations are satisfied provided that the moment of the couple is

$$(2.315) \quad M = \sum_{i=1}^n (x_i Y_i - y_i X_i).$$

To sum up: *In general a plane system of forces is plane-equipollent to a single force whose components and line of action are given by (2.312). If*

coordinates, and let the coordinates of  $A, B$  be  $(x, y, z), (x + \delta x, y + \delta y, z + \delta z)$ , respectively. Then, since the direction cosines of  $\mathbf{P}$  are  $X/P, Y/P, Z/P$  and those of the displacement  $\vec{AB}$  are  $\delta x/\delta s, \delta y/\delta s, \delta z/\delta s$ , we have

$$(2.402) \quad \cos \theta = \frac{X}{P} \frac{\delta x}{\delta s} + \frac{Y}{P} \frac{\delta y}{\delta s} + \frac{Z}{P} \frac{\delta z}{\delta s},$$

and so the work done by  $\mathbf{P}$  in the displacement  $\vec{AB}$  is

$$(2.403) \quad \delta W = X \delta x + Y \delta y + Z \delta z.$$

We note that no work is done when the displacement is perpendicular to the force.

If force is measured in dynamical units, work has dimensions  $[ML^2T^{-2}]$  (see Appendix). In the e.g.s. system, the unit of work is the erg, which is 1 dyne cm. or 1 gm. cm.<sup>2</sup> sec.<sup>-2</sup>; in the f p.s. system, it is 1 ft. poundal or 1 lb. ft.<sup>2</sup> sec.<sup>-2</sup>. In statics, the ft. lb. and ft. ton are used.

Rate of working is called *power*. In dynamical units, power has dimensions  $[ML^2T^{-3}]$ . A unit commonly employed is the horsepower (550 ft. lb. wt. sec.<sup>-1</sup> = 1 hp.).

## Forces which do no work.

Consider a particle in contact with the surface of a fixed rigid body, and suppose that a force acts on the particle tending to drive it into the body. If no other force acted, the particle would have to penetrate the body, in accordance with (1.401). Since, however, we regard such penetration as impossible, we must assume the existence of another force, the *reaction* of the surface, which prevents the penetration from taking place. The particle in question may be an isolated particle, or it may be one of the particles of a rigid body.

In a mechanical problem the reactions between particles and surfaces, or between pairs of surfaces, are not in general to be regarded as known forces. They are called into play solely to prevent violation of the condition of non-penetration.

The words “smooth” and “rough” are familiar in ordinary life; we speak of polished steel, glass, ice, etc., as smooth, and sandpaper, cloth, etc., as rough. We make such a classification primarily on the basis of our sense of touch. A more scientific classification is obtained by examining the directions of the reactions between bodies. It is found that with smooth bodies the reactions always lie very close to the common normal of the surfaces in contact. As an idealization of such bodies, we admit into our mathematical model the concept

of a *smooth surface* with the following property:

*The reaction at a smooth surface is normal to the surface, and is of such a magnitude as just to prevent penetration or overlapping in space from taking place.*

The reaction at a rough surface has more complicated properties which will be discussed in Sec. 3.2.

Since the reaction at a smooth surface is normal to the surface, the following result is evident from (2.401):

*No work is done by the reaction at a smooth fixed surface in an infinitesimal displacement which preserves the contact.*

In any contact there are actually two equal and opposite forces involved, one acting on each body. In what has been said above, one body was supposed fixed, so that the force acting on it did no work. Suppose now that both bodies, having smooth contact, are displaced infinitesimally. The displacements of the particles of the two bodies in contact with one another may now have components along the common normal, and so work is done by each of the two forces of reaction. But it is not difficult to see that *the sum of these two works is zero*.

Let us now consider a rolling contact, confining our attention here, for simplicity, to the rolling of a rigid circle on a fixed line in the fundamental plane (Fig. 20).

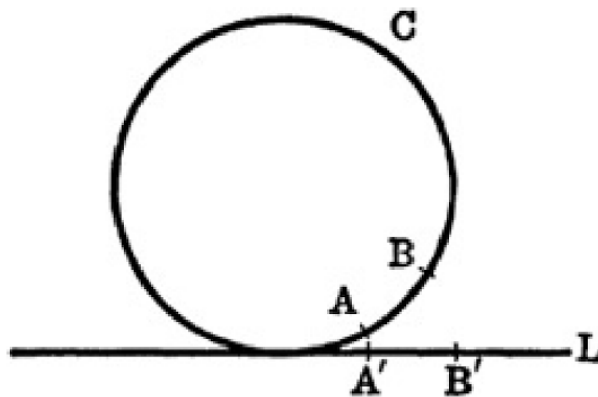


FIG. 20.—A circle rolling on a line.

We say that the circle  $C$  rolls on the line  $L$  if it passes continuously through a sequence of positions such that (i)  $L$  is always tangential to  $C$ ; (ii) if any two points  $A, B$  of  $C$  make contact with points  $A', B'$  of  $L$ , then are  $AB = A'B'$ . (Consider a motor tire and the pattern it leaves on the road.)

If  $C$  advances a distance  $\Delta x$  while it turns through an angle  $\Delta\theta$ , it is clear that

(2.404)

$$\Delta x = a \Delta \theta,$$

where  $a$  is the radius of  $C$ . During this advance the point of  $C$  initially in contact with  $L$  receives the following displacements:

$$\text{Horizontal: } \Delta x - a \sin \Delta \theta,$$

$$\text{Vertical: } a - a \cos \Delta \theta.$$

If the displacement  $\Delta x$  is infinitesimal, then these displacements are infinitesimals of the orders  $(\Delta x)^3$ ,  $(\Delta x)^2$ , respectively. This is easily seen on using the well-known series for sine and cosine.

Suppose now that there is a reaction  $\mathbf{R}$  exerted by  $L$  on  $C$ . We shall not assume that  $\mathbf{R}$  is perpendicular to  $L$ . The work  $\mathbf{W}$  done by  $\mathbf{R}$  in a succession of infinitesimal rolling operations will be a function of  $x$ , the final displacement. It may be written

$$W = \int \frac{dW}{dx} dx.$$

But since the displacement of the point of contact is an infinitesimal of higher order than the increment in  $x$ , we have  $dW/dx = 0$  and hence  $W = 0$ . Thus, *no work is done by the reaction at a rolling contact*. It is true that our proof deals only with the case of a circle rolling on a line; the general case of a moving curve rolling on a fixed curve may be discussed by a slightly more complicated argument leading to the same result. The condition of rolling is the equality of arcs on the two curves between points of contact; the essential point in the proof is the fact that the displacement of the point of the moving curve instantaneously in contact is an infinitesimal of higher order than the infinitesimal angle through which the moving curve turns.

When rolling takes place between two moving surfaces, the sum of the works done by the equal and opposite reactions is zero. This follows from the fact that the displacements of the two particles in contact (one belonging to each body) are equal, to the first order of small quantities.

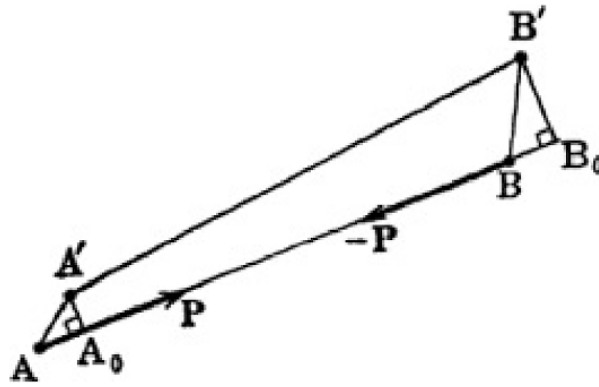


FIG. 21.—Internal reactions in a rigid body.

The fact that no work is done at a rolling contact is of enormous importance in modern transport, which moves on wheels in contrast to the dragged vehicles of primitive civilizations. It explains why boats are launched or hauled out of the water with much greater ease when placed on rollers, and why machinery operates more easily on ball or roller bearings than on plain bearings.

Let us now consider the work done by a pair of equal and opposite reactions, exerted on one another by two particles of a rigid body, when the body receives an infinitesimal displacement. Let  $A, B$  be the positions of the two particles before displacement, and  $A', B'$  their positions after displacement (Fig. 21). The lines  $AB, A'B'$  make an infinitesimal angle with one another, and

$$(2.405) \quad AB = A'B',$$

since the body is rigid. If  $A_0, B_0$  are the projections of  $A', B'$  respectively on the line  $AB$ , we have obviously

$$(2.406) \quad AA_0 + A_0B_0 = AB + BB_0.$$

Since the inclination of  $A'B'$  to  $AB$  is infinitesimal,  $A_0B_0 = A'B'$  to the first order of infinitesimals; hence,  $A_0B_0 = AB$  by (2.405) and so (2.406) gives

$$(2.407) \quad AA_0 = BB_0.$$

If the forces on  $A, B$  are respectively  $\mathbf{P}, -\mathbf{P}$ , the amounts of work done by them are

$$P \cdot AA_0, \quad -P \cdot BB_0,$$

and the sum of these is zero. Hence, *no work is done by a pair of equal and opposite reactions, exerted on one another by two particles of a rigid body.*

To sum up, we have seen that no work is done by



- (i) the reaction on a movable body in smooth contact with a fixed body;
- (ii) the pair of reactions at a smooth contact;
- (iii) the reaction on a body rolling on a fixed body;
- (iv) the pair of reactions at a rolling contact;
- (v) the pair of reactions between two particles of a rigid body.

In the cases considered above, the particles forming the systems are not wholly free. The systems are, in fact, subject to *constraints*, and the reactions are brought into play to prevent the violation of these constraints. Since these *reactions of constraint* do no work in permissible displacements, we speak of these constraints as *workless*.

## The principle of virtual work.

Let us start by considering the simple system shown in Fig. 22.  $AB$  is a rigid bar; a small hole is drilled in it at  $C$ , and a smooth pin passes through the hole, attaching the bar to some fixed support (not shown). Thus the bar can turn about  $C$  in the plane of the paper. Forces  $\mathbf{P}$  and  $\mathbf{Q}$  are applied at  $A$  and  $B$ , respectively, in directions perpendicular to  $AB$ .

There are two ways of regarding this system:

- (i) It is just a rigid body, which can turn round  $C$  and which is subjected to the forces  $\mathbf{P}$  and  $\mathbf{Q}$ .
- (ii) It is a collection of a vast number of particles, subjected not only to the forces  $\mathbf{P}$  and  $\mathbf{Q}$ , but also to a vast number of reactions between the particles and a reaction at  $C$ , all adjusted so that the distances between the particles remain constant and the particles near  $C$  do not move away from the pin.

For present purposes we regard (ii) as the more useful view.

In developing the principle of virtual work, we have to deal with *displacements*, *forces*, and *work*.

The only displacement consistent with the constraints is a rotation around  $C$ . But if we take the point of view (ii) above, we may think of other displacements in which the constraints are violated—for example, only one particle of the bar might be moved from its position. Such a displacement is merely a mathematical device. In either case (whether the constraints are satisfied or not) the displacement is called “virtual,” this word implying that the displacement is a hypothetical one and not a displacement actually experienced. We note then that virtual displacements are of two types:

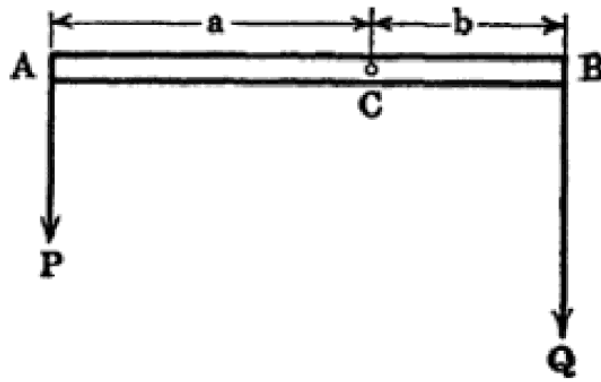


FIG. 22.—A rigid bar pinned at  $C$ .

- (a) virtual displacements satisfying the constraints,
- (b) virtual displacements violating the constraints.

As for forces, we have  $\mathbf{P}$  and  $\mathbf{Q}$  and the reactions of constraint. We need a word to distinguish  $\mathbf{P}$  and  $\mathbf{Q}$  from the latter; we cannot use the word “external” because the reaction exerted by the pin at  $C$  is external. So we shall call  $\mathbf{P}$  and  $\mathbf{Q}$  *applied* forces, with this general definition for any system with workless constraints: *Forces other than reactions of constraint are called applied forces.* Thus the forces acting on the system are of two types:

- (a) applied forces,
- (b) reactions of constraint.

As for the work done in a virtual displacement (called *virtual work*), we are to observe that no work is done by the reactions of constraint provided that the constraints are of the workless type and are satisfied by the virtual displacement.

We shall now proceed to state and prove the principle of virtual work. The argument will be quite general, covering the case of any system in which the constraints are of the workless type.

**PRINCIPLE OF VIRTUAL WORK.** *A system with workless constraints is in equilibrium under applied forces if, and only if, zero virtual work is done by the applied forces in an arbitrary infinitesimal displacement satisfying the constraints.*

It will be noticed that this statement contains both a sufficient (if) condition for equilibrium and a necessary (only if) condition. There are two theorems here, requiring separate proofs.

Let us first prove the *necessity* of the condition; that is, we are given that the system is in equilibrium, and we have to prove that zero virtual work is done by the applied forces in any infinitesimal displacement satisfying the constraints. Consider any particle of the system; it is in equilibrium, and so the resultant of all forces acting on it is zero. Thus, zero virtual work is done by the

forces acting on that particle in any displacement of it. This holds for all particles; and so, in any displacement of the system, zero virtual work is done by *all* forces acting. But the reactions of constraint do no work in any displacement which satisfies the constraints. Hence the applied forces do no work in such a displacement, and so the necessity of the condition is proved.

Let us now prove its *sufficiency*; that is, we are given that zero virtual work is done by the applied forces in any infinitesimal displacement satisfying the constraints, and we have to prove that the system is in equilibrium. *Suppose it is not in equilibrium*; then it starts to move. It is clear from (1.401) that each particle starts to move in the direction of the resultant force acting on it. Referring to (2.401), we see that a *positive* amount of work is done in the initial displacement, since  $\theta = 0$  and  $\cos \theta = 1$ . This is true for every particle; and so, in the initial displacement of the system, positive (not zero) virtual work is done by all the forces. But such an initial displacement must of course satisfy the constraints, and so the reactions of constraint do no work. Thus, on the basis of our assumption that the system is not in equilibrium, we see that it must undergo a displacement which satisfies the constraints and in which the applied forces do positive work. But this contradicts the given information, according to which zero work is done. Since our assumption leads to a contradiction, it must be false, and so the system does remain in equilibrium. The sufficiency of the condition is proved.

Returning to the system shown in Fig. 22, let us give the bar a rotation about  $C$  in a counterclockwise sense through an infinitesimal angle  $\delta\theta$ . The work done by  $\mathbf{P}$  is  $Pa \delta\theta$  and the work done by  $\mathbf{Q}$  is  $-Qb \delta\theta$ , and so the total work is

$$\delta W = (Pa - Qb) \delta\theta.$$

If the system is in equilibrium, then  $\delta W = 0$  and hence

$$\frac{P}{Q} = \frac{b}{a}.$$

On the other hand, if  $P/Q = b/a$ , then  $\delta W = 0$  for this displacement. But this is the most general displacement satisfying the constraints; hence the bar must be in equilibrium if the condition  $P/Q = b/a$  is satisfied.

Although the chief merit of the principle of virtual work lies in the fact that it does not involve the reactions of constraint, nevertheless it can be used to find these reactions should they be required. Suppose, for example, we wish to know the reaction at  $C$  in the system considered. If this reaction is  $\mathbf{R}$ , it is obvious that the equilibrium will not be disturbed if we remove the pin at  $C$

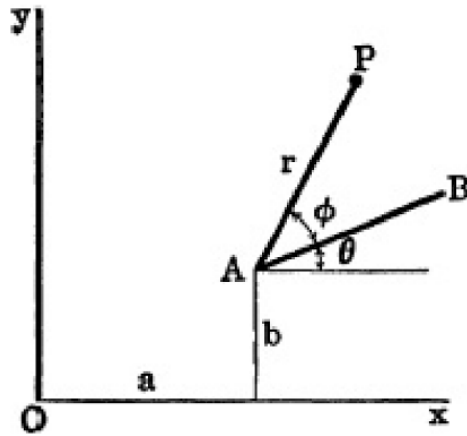


FIG. 24.—In an infinitesimal displacement the quantities  $a$ ,  $b$ , and  $\theta$  receive infinitesimal increments, while  $r$  and  $\phi$  remain constant. In the fundamental plane instead of displacements of the rigid body parallel to that plane; they come to the same thing.

In Fig. 23,  $L$  is the lamina before and  $L'$  the lamina after an arbitrary displacement. Let  $A$ ,  $B$  be any two points of the lamina before displacement and  $A'$ ,  $B'$  their positions after displacement. Obviously the displacement from  $L$  to  $L'$  may be achieved in two steps:

(i) a *translation*, in which each point of  $L$  receives a displacement equal and parallel to  $\vec{AA'}$ ;

(ii) a *rotation* about  $A'$  through an angle equal to the angle between  $AB$  and  $A'B'$ .

The point  $A$ , used in describing the displacement, is called a *base point*.

Let us now consider an *infinitesimal displacement*. For fixed axes  $Oxy$  in the fundamental plane (Fig. 24), let  $(a, b)$  be the coordinates of  $A$ , and  $\theta$  the inclination of  $AB$  to  $Ox$ . Then the increments  $\delta a$ ,  $\delta b$  describe the translational displacement, and the increment  $\delta\theta$  describes the rotation. Let  $P$  be any particle of the lamina; let us put  $r = AP$ ,  $\phi = \angle BAP$ . Since the lamina is rigid,  $r$  and  $\phi$  remain constant as the lamina moves.

Now if  $(x, y)$  are the coordinates of  $P$ , we have

$$(2.408) \quad x = a + r \cos (\theta + \phi), \quad y = b + r \sin (\theta + \phi).$$

Hence the displacement of  $P$  in the infinitesimal displacement  $\delta a$ ,  $\delta b$ ,  $\delta\theta$  of the lamina is

$$\delta x = \delta a - r \sin (\theta + \phi) \delta\theta, \quad \delta y = \delta b + r \cos (\theta + \phi) \delta\theta.$$

Substituting for  $r \sin (\theta + \phi)$ ,  $r \cos (\theta + \phi)$  from (2.408), we obtain

$$(2.409) \quad \delta x = \delta a - (y - b) \delta\theta, \quad \delta y = \delta b + (x - a) \delta\theta.$$

This gives the infinitesimal displacement of any point in the lamina (or in the

rigid body of which the lamina is a section) in terms of the translation  $(\delta a, \delta b)$  of the base point  $(a, b)$ , and the rotation  $\delta\theta$ . By giving arbitrary infinitesimal values to  $\delta a, \delta b, \delta\theta$ , we get the most general infinitesimal displacement of a lamina in a plane or of a rigid body parallel to a plane.

*Exercise.* Find the displacement of the particle at the highest point of a rolling wheel, when the wheel advances a small distance  $\delta s$ .

### **Sufficient conditions for the equilibrium of a rigid body movable parallel to a fixed plane.**

We now consider the equilibrium of a rigid body which can move only parallel to a fixed fundamental plane. There are no constraints limiting the motion of the body parallel to the plane, but there are constraints preventing other motions. These are supposed to be of the workless type. In addition to the reactions of these constraints, there act applied forces, not necessarily parallel to the fundamental plane. Let  $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$  be the projections on the fundamental plane ( $z = 0$ ) of the points at which these forces are applied, and let  $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$  be the components of these forces in the directions of the axes  $Oxy$ .

In an infinitesimal virtual displacement  $\delta a, \delta b, \delta\theta$ , as described above, the work done is

$$\begin{aligned}
 (2.410) \quad \delta W &= \sum_{i=1}^n X_i [\delta a - (y_i - b) \delta\theta] \\
 &\quad + \sum_{i=1}^n Y_i [\delta b + (x_i - a) \delta\theta] \\
 &= \left( \sum_{i=1}^n X_i \right) \delta a + \left( \sum_{i=1}^n Y_i \right) \delta b \\
 &\quad + \sum_{i=1}^n [(x_i - a) Y_i - (y_i - b) X_i] \delta\theta \\
 &= X \delta a + Y \delta b + N \delta\theta,
 \end{aligned}$$

where  $X, Y$  are the components of the vector sum of the applied forces and  $N$  is their moment about the point  $(a, b)$ .

Thus, if

$$(2.411) \quad X = 0, \quad Y = 0, \quad N = 0,$$

we have  $\delta W = 0$  for the most general infinitesimal displacement consistent with the constraints. Thus, by the principle of virtual work, the body is in



equilibrium if (2.411) are satisfied. These are therefore sufficient conditions for the equilibrium of the body. Let us restate this important result, as follows:

*If a rigid body is constrained to move parallel to a fixed fundamental plane, then the body is in equilibrium under the action of any system of external forces plane-equipollent to zero; i.e., there is equilibrium provided that the vector sum of the projections of these forces on the fundamental plane vanishes, and the moment of these forces about some one line perpendicular to the fundamental plane vanishes also.*

We note that (2.411) are the same as (2.306) or (2.308), written in a slightly different form. In Sec. 2.3 these conditions were shown to be *necessary* for the equilibrium of any system; now we find them to be *sufficient* for the equilibrium of a rigid body movable parallel to the plane  $z = 0$ .

Let  $S$  and  $S'$  be two plane-equipollent force-systems acting on a rigid body; then  $X$ ,  $Y$ ,  $N$  have the same values for  $S$  and  $S'$ . It follows from (2.410) that, if  $\delta W$  is the virtual work done by  $S$  and  $\delta W'$  that done by  $S'$  in the *same displacement* of the body parallel to the fundamental plane, then

$$\delta W = \delta W'.$$

It is evident that two plane-equipollent force systems are equivalent in all statical problems concerning the equilibrium of a rigid body movable parallel to the fundamental plane, in the sense that one system may be replaced by a plane-equipollent system without disturbing equilibrium. In particular, two forces are equivalent if they are equal in magnitude, with the same sense and with a common line of action. Thus, we may slide a force along its line of action without changing its effect. This is known as the principle of *transmissibility of force* for a rigid body, and it enables us to regard a force acting on a rigid body as a sliding vector.

In the same way, two couples in the same plane are equivalent if they have the same moment  $N$ . Since  $X = Y = 0$  for a couple, it follows from (2.410) that *the work done by a couple of moment  $N$  in an infinitesimal rotation  $\delta\theta$  is  $N \delta\theta$ .*

There is no unique way in which the principles of mechanics must be developed. Two rival methods exist, the method of forces as used in Sec. 2.3 and the method of virtual work as used here. In developing the theory up to this stage, we have used one method to establish some points and the other method to establish other points. The reader may prefer to work out some other logical development of the subject, and indeed is most likely to appreciate the critical points in the chain of reasoning by so doing.

*Exercise.* A card lies on a table. Along the edges, there are applied forces represented in magnitude and direction by the edges, taken in order. The card is

given any small displacement. Show that the work done is represented by twice the area of the card, multiplied by the angle of rotation.

### Potential energy.

Consider any system of particles. We shall refer to a set of positions of all the particles as a *configuration* of the system. Let  $A_0$  be some configuration selected as a *standard configuration*, and let  $A$  be any other configuration. Let us take the system from  $A$  to  $A_0$ , and denote by  $W$  the work done by all forces acting on the system during this process.

If the system consists of a single particle in the plane  $Oxy$ , we might take the origin  $O$  as standard configuration  $A_0$ . Then if  $X, Y$  are the components of force acting on it, the work done in bringing it from the position  $A$  is

$$(2.412) \quad W = \int_A^O (X dx + Y dy),$$

the integral being taken along the curve by which the particle is brought to the origin. For example, if the force depends on the position of the particle according to the equations  $X = 2x, Y = 6y$  and if the coordinates of  $A$  are  $a, b$ , then

$$(2.413a) \quad W = \int_{a,b}^{0,0} (2x dx + 6y dy) = \left[ x^2 + 3y^2 \right]_{a,b}^{0,0} = -a^2 - 3b^2.$$

This value is independent of the particular path along which the particle is brought to  $O$ .

To take another example, if  $X = 6y, Y = 2x$ , we have

$$(2.413b) \quad W = \int_{a,b}^{0,0} (6y dx + 2x dy).$$

The value of this integral is not independent of the path of integration, as is easily seen by taking the two paths along the sides of the rectangle  $x = 0, y = 0, x = a, y = b$ . Thus it is only in some cases that  $W$ , is independent of the path.

Passing from the case of a single particle to a general system, we make the following definition:

When the forces acting on a system are such that the work done by them, in the passage of the system from a configuration  $A$  to the standard configuration  $A_0$  is independent of the way in which this passage is carried out, then the system is said to be *conservative*. The work done by the forces in the passage

from  $A$  to  $A_0$  is called the *potential energy*\* of the system at the configuration  $A$ .

Since the standard configuration may be chosen arbitrarily, the potential energy of a conservative system is indeterminate to within an additive constant, the value of which depends on the standard configuration chosen.

Potential energy will be denoted by  $V$ . Thus, in the case of (2.413a), the system (a single particle) is conservative and the potential energy  $V$  at the position  $x, y$  is

$$(2.414) \quad V = -x^2 - 3y^2.$$

Generally speaking, most of the systems considered in mechanics are conservative. The outstanding exceptions are systems in which frictional resistances are involved. Cases like that of (2.413b) occur rarely.

Let there be a conservative system,  $A_0$  being the standard configuration. We shall use the following notation:

$$\begin{aligned} V(A) &= \text{potential energy at configuration } A, \\ W(A, B) &= \text{work done by forces in passage from } A \text{ to } B. \end{aligned}$$

By the definition of  $V$ , we have

$$(2.415) \quad V(A) = W(A, A_0).$$

Now give the system an infinitesimal displacement from  $A$  to  $B$ ; let  $\delta V$  be the increment in potential energy and  $\delta W$  the work done. We have

$$\begin{aligned} \delta V &= V(B) - V(A) \\ &= W(B, A_0) - W(A, A_0). \end{aligned}$$

But since work done is independent of path and additive, we have

$$W(B, A_0) = W(B, A) + W(A, A_0),$$

and

$$W(B, A) = -W(A, B) = -\delta W.$$

Hence we have

$$(2.416) \quad \delta V = -\delta W.$$

In words, *the increment in potential energy equals the work done, with its sign changed.*

Consider a particle which can move in a plane subject to the action of a force which depends only on the position of the particle. If  $X, Y$  are the components of force and  $x, y$  the coordinates of the particle, then  $X, Y$  are functions of  $x, y$ . This is called a *field of force*. It is said to be a *conservative field* if the particle under its influence is a conservative system. In that case, denoting the potential energy by  $V$ , we have by (2.403) and (2.416) the equation

$$(2.417) \quad X \delta x + Y \delta y = -\delta V,$$

where  $\delta x, \delta y$  are the components of an arbitrary infinitesimal displacement given to the particle. It follows that

$$(2.418) \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}.$$

The preceding result may be extended without any difficulty to a conservative field in space; we have then

$$(2.419) \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}.$$

In the language of Sec. 1.3: *In a conservative field, the force is the gradient of potential energy, with sign reversed.*

A *uniform* field of force is one in which  $X, Y, Z$  are constants. Such a field is conservative, with potential energy

$$(2.420) \quad V = -(Xx + Yy + Zz).$$

Returning to a general conservative system, let us note a useful consequence of (2.416) in connection with the principle of virtual work. *If a conservative system is in equilibrium, the change in potential energy in any infinitesimal displacement is zero.* This is also expressed by saying that the potential energy has a *stationary value*.

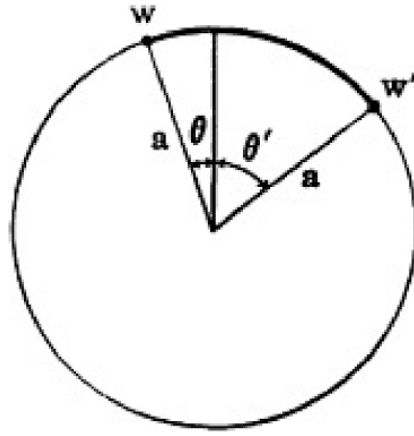


FIG. 25.—Two heavy particles balanced on a smooth cylinder.

*Example.* As an illustrative example of the principle of virtual work, consider the system shown in Fig. 25. Two heavy particles of weights  $w$ ,  $w'$  are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radius  $a$ , the axis of which is horizontal. We wish to find the position of equilibrium.

Here we have a system of two particles. The forces acting on them are

- (i) gravity,
- (ii) forces due to the tension in the string,
- (iii) reactions exerted by the cylinder.

If we give a virtual displacement satisfying the constraints, only gravity does work. The system is conservative, and the potential energy is, by (2.420), with suitable choice of the standard configuration,

$$V = wa \cos \theta + w'a \cos \theta',$$

where  $\theta$ ,  $\theta'$  are the inclinations to the vertical of the radii drawn to the particles. In an infinitesimal displacement,

$$\delta V = -wa \sin \theta \delta \theta - w'a \sin \theta' \delta \theta'.$$

But  $\theta + \theta'$  is constant, since the string is inextensible. Thus  $\delta \theta' = -\delta \theta$ , and

$$\delta V = a \delta \theta (w' \sin \theta' - w \sin \theta).$$

Hence, when the system is in equilibrium, the following condition must be satisfied:

$$\frac{\sin \theta}{\sin \theta'} = \frac{w'}{w}.$$



## 2.5. STATICALLY INDETERMINATE PROBLEMS

Consider a rigid body which can move parallel to a fixed fundamental plane, in equilibrium under external forces in that plane. Let  $X$  and  $Y$  be the total components of the external forces on axes of coordinates in the fundamental plane and  $N$  their total moment about the origin. Then, as in (2.411), we have the scalar equations of equilibrium

$$(2.501) \quad X = 0, \quad Y = 0, \quad N = 0.$$

It is important to note that these equations are *three* in number. This means that, in any problem concerning the plane statics of a rigid body, we can find *three* unknowns and no more. If there are more than three unknowns, the problem is *statically indeterminate*, which means that it cannot be solved by means of the conditions of equilibrium alone. We shall illustrate with a simple example.

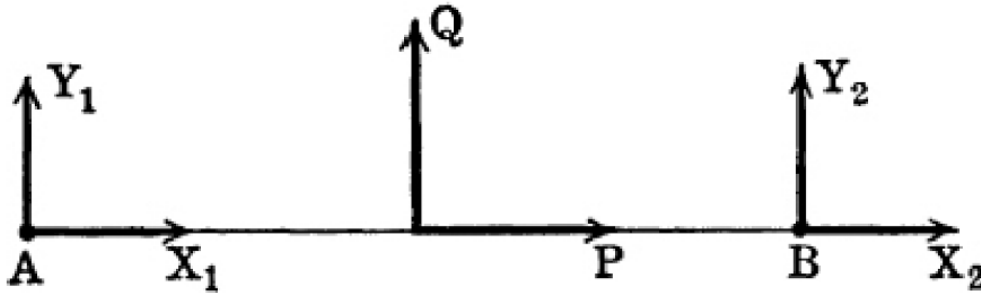


FIG. 26.—A statically indeterminate problem.

A rigid bar  $AB$  (Fig. 26) is fixed at its ends; at its middle point there is applied a force with components  $P$ ,  $Q$  along and perpendicular to the bar. Find the reactions on the bar at  $A$  and  $B$ .

Let the components of the reactions be  $X_1$ ,  $Y_1$  at  $A$  and  $X_2$ ,  $Y_2$  at  $B$ , and let  $AB = 2a$ . Taking components along and perpendicular to the bar and moments about  $A$ , by (2.501) we have

$$(2.502) \quad \begin{cases} X = X_1 + P + X_2 = 0, \\ Y = Y_1 + Q + Y_2 = 0, \\ N = aQ + 2aY_2 = 0. \end{cases}$$

Hence,

$$(2.503) \quad Y_2 = -\frac{1}{2}Q, \quad Y_1 = -\frac{1}{2}Q, \quad X_1 + X_2 = -P.$$

We have only *three* equations for *four* unknowns; the problem is statically

indeterminate, and nothing more can be found out about the reactions from the conditions of equilibrium alone.

If such a problem were to occur in physical reality, the four unknown quantities would have values which might be measured. Apparently our mathematical methods have failed us; they have not provided us with the answer to a question of physical interest. The fault actually lies in the selection of a mathematical model. Rigid bodies do not exist in nature, and this problem is one where the use of a rigid body as a mathematical model is not justified. We should take an elastic bar as a model instead (cf. Sec. 3.3).

A simple modification in the constraints may render an indeterminate problem determinate. Consider the same problem, altered by the condition that the end  $B$ , instead of being

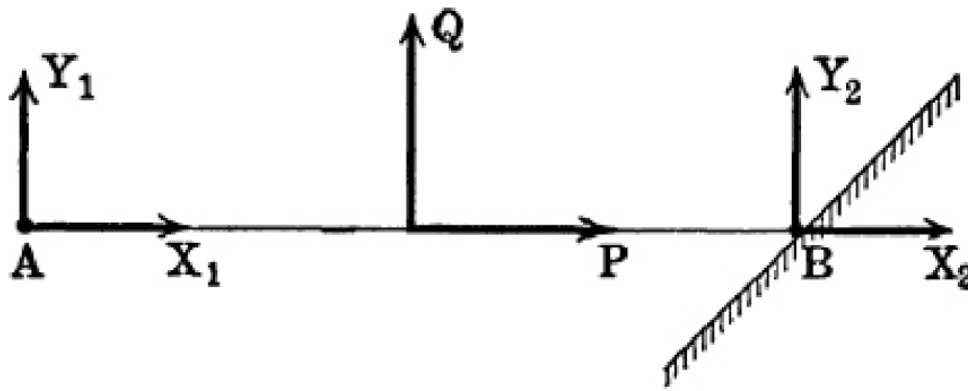


FIG. 27.—A statically determinate problem.

fixed, slides on a smooth plane inclined at an angle of  $45^\circ$  to  $AB$  (Fig. 27). Again, we have (2.503), but also an additional equation

$$(2.504) \quad X_2 + Y_2 = 0,$$

arising from the condition that the reaction at  $B$  is perpendicular to the plane. Now the problem is statically determinate, and we have

$$(2.505) \quad \begin{cases} X_1 = -P - \frac{1}{2}Q, & Y_1 = -\frac{1}{2}Q, \\ X_2 = \frac{1}{2}Q, & Y_2 = -\frac{1}{2}Q. \end{cases}$$

Had we taken one of the axes parallel to the plane of constraint at  $B$ , we should have obtained a slightly simpler treatment, because only three unknowns would have appeared.

## 2.6. SUMMARY OF METHODS OF PLANE STATICS

### I. Conditions of equilibrium.

(a) For single particle (necessary and sufficient):

**(2.601) Vector sum of all forces vanishes,**

or

**(2.602) Total components in two perpendicular directions vanish**

(b) For any system (necessary) or for rigid body (necessary and sufficient):

**(2.603)  $\mathbf{F} = 0$ ,  $N = 0$ ; or  $X = 0$ ,  $Y = 0$ ,  $N = 0$ .**

(c) For any system with workless constraints (necessary and sufficient):

**(2.604)  $\delta W = 0$  (work done by applied forces).**

## **II. Moment of a vector in a plane about a point in that plane.**

**(2.605)  $M = \pm aP$  (+ for counterclockwise);**

**(2.606)  $M = xY - yX$  (about origin).**

## **III. Plane equipollence.**

(a) Conditions for plane equipollence:

**(2.607)  $\mathbf{F} = \mathbf{F}'$ ,  $N = N'$ .**

(b) General system of forces can be reduced to a single force at an assigned point, together with a couple.

(c) General system of forces can be reduced to a single force or to a single couple (latter case exceptional).

## **IV. Work and potential energy.**

(a) Definition of work:

**(2.608)  $\delta W = P \cos \theta \cdot \delta s = X \delta x + Y \delta y$ .**

(b) Reactions do no work at smooth contacts, rolling contacts, and inside rigid body.

(c) For the infinitesimal displacement of a rigid body,

**(2.609)  $\delta W = X \delta a + Y \delta b + N \delta \theta$ .**

(d) Potential energy:

$$(2.610) \quad \delta V = -\delta W,$$

$$(2.611) \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}.$$

(Force is gradient of potential energy with sign reversed.)

## EXERCISES II

The method of virtual work may be used in any of these problems; it will be found particularly useful in the case of those marked with an asterisk.

1. A particle is in equilibrium under the action of six forces. Three of these forces are reversed, and the particle remains in equilibrium. Prove that it will still remain in equilibrium if these three forces are removed altogether.

2. A ladder of weight  $W$  rests at an angle  $I'$  to the horizontal, with its ends resting on a smooth floor and against a smooth vertical wall. The lower end is joined by a rope to the junction of the wall and the floor. Find, in terms of  $W$  and  $\alpha$ , the tension of the rope and the reactions at the wall and the ground. (Assume that the weight of the ladder acts at its middle point.)

3. The corner  $A$  of a square plate  $ABCD$  is held fixed by means of a smooth hinge which permits the plate to turn freely in its own plane. Four forces, each of magnitude  $P$ , act along the four sides in order. Find the single additional force which, applied at the center of the plate parallel to the side  $AB$ , will keep the plate in equilibrium. What is the corresponding reaction at the hinge?

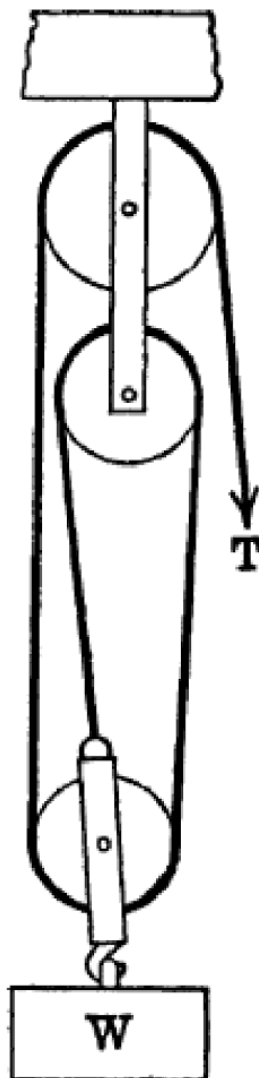
4. A door of weight  $W$ , height  $2a$ , and width  $2b$  is hinged at the top and bottom. If the reaction at the upper hinge has no vertical component, find the components of reaction at both hinges. (Assume that the weight of the door acts at its center.)

5. A particle of weight  $W$  is suspended from a fixed point by a light string. A horizontal force  $H$  is applied to it, and the particle takes up a position of equilibrium with the string inclined to the vertical. If the string breaks when the tension in it reaches a value  $T_0$ , find the smallest value of  $H$  necessary to break the string.

6. A heavy beam  $AB$ , 8 ft. long, rests horizontally on two supports, one at  $A$  and the other 3 ft. from  $B$ . If the greatest weight that can be hung from  $B$  without upsetting the beam is 20 lb., find the weight of the beam. (Assume that the weight of the beam acts at its middle point.)

7. Show that if a light cable passes round a pulley mounted on smooth

bearings, the tensions in the portions on either side of the pulley are equal. Hence find the tension  $T$  in the cable for the pulley system shown, supporting a weight  $W$ , the pulleys and cable being supposed light and the distance between the upper and lower pulleys so great that the cables may be regarded as vertical.



8. A force of magnitude  $P$ , acting up and along a smooth inclined plane, can support a weight  $W$ ; when acting horizontally, it can support a weight  $w$ . Find a relation among  $P$ ,  $W$ , and  $w$ , not involving the inclination of the plane.

9. Four lamps each weighing 4 lb. are suspended across a road between posts 40 ft. apart by light cords attached at points  $B$ ,  $C$ ,  $D$ ,  $E$  of a cord  $ABCDEF$ , whose ends  $A$  and  $F$  are fixed at the same level to the posts. The cords supporting the lamps divide the horizontal distance between the posts into equal parts.  $C$  and  $D$  are 12 ft. below  $AF$ . Find the tension in  $CD$ .

\*10. A light rigid rod of length 26, terminated by heavy particles of weights  $w$ ,  $W$ , is placed inside a smooth hemispherical bowl of radius  $a$ , which is fixed with its rim horizontal. If the particle of weight  $w$  rests just below the rim of the bowl, prove that



$$wa^2 = W(2b^2 - a^2).$$

**11.** A system of forces acting on a rigid body consists of  $n$  forces acting along the  $n$  sides of a closed polygon taken in order. If the magnitudes of the forces are proportional to the lengths of the sides along which they act, show that the system reduces to a couple whose moment is proportional to the area enclosed by the polygon, a proper convention as regards the sign of this area being made. Give a simple example of such a system of forces which would keep a rigid body in equilibrium.

**12.** Explain why in a motion picture the spokes of a rotating wheel sometimes appear to be moving the wrong way.

**13.** A force  $\mathbf{P}$  of constant magnitude and fixed direction is applied to one end of an arm of length  $a$ , which can turn about the other end in a plane containing the direction of  $\mathbf{P}$ . Find the total work done by the force as it pulls the arm into its own direction from a position perpendicular to it.

**14.** Show that a field of force with components  $(X, Y)$  is conservative if, and only if,

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

**15.** Find the potential energy of a particle attracted toward a fixed point by a force of magnitude  $k^2/r^n$ ,  $r$  being the distance from the fixed point and  $k, n$  any constants.

**\*16.** A light lever, in the form of a letter L with arms  $a$  and  $b$ , is pivoted at the angle so that it can turn freely in a vertical plane. Weights  $W, w$  are suspended from the ends. Show that there are just two positions of equilibrium.

**\*17.** To a number of fixed points  $A_1, A_2, \dots, A_n$ , situated at equal intervals  $a$  on a straight line inclined at an angle  $\theta$  to the horizontal, there are attached rods all of the same length  $a$  and weight  $w$ . The other ends of these rods,  $B_1, B_2, \dots, B_n$ , are connected by rods of the same length  $a$  and weight  $w$ . The system hangs in a vertical plane, forming a set of squares,  $A_1$  and  $B_2$  being connected by a light rigid rod. Find the reaction in this rod, assuming that all the joints are smooth and that the weight of each rod acts at its middle point.

**\*18.** A framework  $ABCD$  consists of four equal, light rods smoothly jointed together to form a square; it is suspended from a peg at  $A$ , and a weight  $W$  is attached to  $C$ , the framework being kept in shape by a light rod connecting  $B$  and  $D$ . Determine the thrust in this rod.

**19.** A number of coplanar forces act on a rigid body. All the forces are

turned in their plane through the same angle  $\theta$  about their points of application, without change of magnitude. Show that their resultant turns through the angle  $\theta$  about a fixed point in the body. (This point is called the *astatic center*.)

**20.** Four forces of magnitudes 1, 3, 4, 6 act in order along the sides of a square  $ABCD$  of side  $a$ , the force of magnitude 1 acting along  $AB$ . Choosing as axes in the plane the lines  $AB$  and  $AD$ , find the equation of the line of action of the resultant force. Find also the position of the astatic center (see Exercise 19), if one force only is considered as acting through each corner of the square and the force of magnitude 1 acts at  $A$ .

\* For simplicity, we have taken the case where  $M$  is positive; the other case is dealt with similarly.

\* The word *equivalent* is often used. But there is a danger of confusion in using a common word in a technical sense. We might think that the effects of two such force systems were the same; this is not always the case. If we pull a string with equal and opposite forces at its ends, we produce a very different effect from that caused by pushing it with these forces reversed in direction; yet the two force systems are equipollent.

\* That is, the moments about a line perpendicular to the fundamental plane, cutting it at  $O$  (see p. 44).

\* The use of the symbol  $\delta$  instead of the more usual  $d$  (for differential) is traditional, and not of much importance as far as statics is concerned. But in dynamics it is necessary to distinguish between a purely hypothetical (or virtual) displacement  $\delta x$  and the displacement  $dx$  actually occurring in time  $dt$  (i e.,  $dx = \dot{x} dt$ ).

\* Since potential energy is defined as work done, it has the same dimensions  $[ML^2T^{-2}]$  and is measured in the same units as work (cf. p. 54).

## CHAPTER III

### APPLICATIONS IN PLANE STATICS

In this chapter we shall be concerned chiefly with systems lying in a plane. However, mass centers and centers of gravity are here discussed for systems in space; the presence of a third coordinate causes no real complication.

#### 3.1. MASS CENTERS AND CENTERS OF GRAVITY

##### Definition of mass center.

Consider a system of  $n$  particles of masses  $m_1, m_2, \dots, m_n$ , situated at points  $P_1, P_2, \dots, P_n$ . If the position vectors of these points relative to some assigned point are  $r_1, r_2, \dots, r_n$ , we define the *linear moment* of the system with respect to that point to be the vector

$$\sum_{i=1}^n m_i \mathbf{r}_i.$$

The *mass center* of the system is defined to be that point with respect to which the linear moment vanishes. To show that this definition is significant, we have to prove two things: (i) a mass center exists; (ii) there is only one mass center.

To establish the existence of a mass center, we take any point  $O$ ; let the position vectors of  $P_1, P_2, \dots, P_n$  relative to  $O$  be  $r_1, r_2, \dots, r_n$ . Let  $C$  be the point such that

$$(3.101) \quad \vec{OC} = \frac{\sum_{i=1}^n m_i \mathbf{r}_i}{\sum_{i=1}^n m_i}.$$

Then the position vector of the point  $P_i$  relative to  $C$  is

$$\mathbf{r}_i - \vec{OC},$$

and so the linear moment of the system with respect to  $C$  is

$$\sum_{i=1}^n m_i(\mathbf{r}_i - \vec{OC}) = \sum_{i=1}^n m_i \mathbf{r}_i - \vec{OC} \sum_{i=1}^n m_i.$$

But this vanishes by (3.101), and therefore  $C$  is a mass center.

To establish the uniqueness of the mass center, we assume that there are two mass centers  $C, C'$ , relative to which the position vectors of the particles are  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$  and  $\mathbf{r}'_1, \mathbf{r}'_2, \dots, \mathbf{r}'_n$ , respectively. Then,

$$(3.102) \quad \sum_{i=1}^n m_i \mathbf{r}_i = \mathbf{0}, \quad \sum_{i=1}^n m_i \mathbf{r}'_i = \mathbf{0}.$$

But

$$\mathbf{r}_i = \mathbf{r}'_i + \vec{CC'};$$

combined with (3.102), this leads to  $\vec{CC'} = \mathbf{0}$ , so that  $C$  and  $C'$  coincide.

Equation (3.101) gives the position vector of the mass center relative to an arbitrary origin  $O$ ; *this position vector is the quotient of the linear moment by the total mass*. It follows that the linear moment of a system is the same as that of a particle, having a mass equal to the total mass of the system, situated at its mass center.

If the system consists of only two particles, with masses  $m_1, m_2$ , the definition shows that the mass center lies on the line joining them and divides it in the ratio  $m_2:m_1$ .

For the calculation of mass centers, it is convenient to have (3.101) in scalar form; referred to any axes, the coordinates of the mass center are

$$(3.103) \quad \bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}, \quad \bar{z} = \frac{\sum_{i=1}^n m_i z_i}{\sum_{i=1}^n m_i}.$$

The numerators are, of course, the components of the linear moment with respect to the origin.

It is important to note that when we move a system of particles rigidly (i.e., without changing mutual distances), the mass center is carried along as if rigidly attached to the system. This follows from (3.103). For let  $Oxyz$  be any set of axes and  $O'x'y'z'$  a new set of axes, such that the new position of the system relative to  $O'x'y'z'$  is the same as the old position relative to  $Oxyz$ ; this means that  $x'_i = x_i, y'_i = y_i, z'_i = z_i$ . Then the coordinates of the new mass center relative to  $O'x'y'z'$  will be the same three numbers as the coordinates of the

old mass center relative to  $Oxyz$ , and hence the new mass center occupies the same position relative to the system as the old one did.

Let us now consider a continuous distribution of matter instead of a system of particles. Viewing the continuous distribution as the limit of the discontinuous system, we are led to associate a definite mass with any volume in the continuous distribution. *Density* is defined as mass per unit volume; by this we mean that the density  $\rho$  is

$$(3.104) \quad \rho = \lim \frac{\Delta m}{\Delta v},$$

where  $\Delta m$  is the mass in the volume  $\Delta v$  and the sign “lim” means “limit as  $\Delta v$  contracts to a point.” In an infinitesimal volume  $dv$  the mass is

$$(3.105) \quad dm = \rho dv.$$

In *homogeneous* bodies (with which we shall be chiefly concerned),  $\rho$  is a constant. If  $\rho$  varies from point to point in a body, the body is said to be *heterogeneous*.

The definition given above for the linear moment of a discontinuous system suggests that the linear moment of a continuous system should be defined as

$$\iiint \mathbf{r} \rho \, dx \, dy \, dz,$$

where  $\mathbf{r}$  is the position vector of a general point of the system and  $\rho$  the density at that point. This vector has components

$$\iiint x \rho \, dx \, dy \, dz, \quad \iiint y \rho \, dx \, dy \, dz, \quad \iiint z \rho \, dx \, dy \, dz.$$

The previous definition of mass center leads us to the statement that the mass center is that point for which, taken as origin, we have

$$(3.106) \quad \iiint x \rho \, dx \, dy \, dz = 0, \quad \iiint y \rho \, dx \, dy \, dz = 0, \\ \iiint z \rho \, dx \, dy \, dz = 0.$$

To find the mass center we may use (3.103), changed into continuous form. Thus, for any axes, the mass center has coordinates

$$(3.107) \quad \bar{x} = \frac{\iiint x \rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \quad \bar{y} = \frac{\iiint y \rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \\ \bar{z} = \frac{\iiint z \rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}.$$

Consideration of a system of particles lying in or very close to a plane or surface leads to the idealized concept of a continuous distribution of matter on a plane or surface; we introduce a quantity  $\sigma$  called *surface density*, such that the



mass of an infinitesimal area  $dS$  of the surface is  $\sigma dS$ . The mass center of a surface distribution has coordinates

$$(3.108) \quad \bar{x} = \frac{\iint x\sigma dS}{\iint \sigma dS}, \quad \bar{y} = \frac{\iint y\sigma dS}{\iint \sigma dS}, \quad \bar{z} = \frac{\iint z\sigma dS}{\iint \sigma dS}.$$

Similarly, we consider a continuous distribution of matter along a line or curve; we introduce a quantity  $\lambda$  called the *line density*, such that the mass of an element  $ds$  is  $\lambda ds$ . The mass center of a curvilinear distribution has coordinates

$$(3.109) \quad \bar{x} = \frac{\int x\lambda ds}{\int \lambda ds}, \quad \bar{y} = \frac{\int y\lambda ds}{\int \lambda ds}, \quad \bar{z} = \frac{\int z\lambda ds}{\int \lambda ds}.$$

In (3.107), (3.108), and (3.109) the denominator in each case represents the total mass of the system.

In the case of uniform distributions of mass (i.e., distributions of constant volume density, surface density, or line density, as the case may be), the density factor comes outside the signs of integration and so disappears by cancellation from (3.107), (3.108), and (3.109).

## Methods of symmetry and decomposition.

In cases of symmetry, it is possible to locate the mass center (or, at any rate, limit its position) without any calculation.

A system is said to have *central symmetry* with respect to a point  $O$  if the system is left unchanged by reflection in the point  $O$ . (By reflection we mean that a particle or element of mass  $m$  at  $A$  is replaced by a particle or element of mass  $m$  at  $B$ , where  $\vec{OB} = -\vec{OA}$ .) For such a system, it is immediately seen that the mass center coincides with the center of symmetry, because the linear moment about that point consists of contributions which cancel in pairs.

A system has a *plane of symmetry* if the system is left unchanged by reflection in a plane. It is easily seen that in such cases the mass center lies in the plane of symmetry.

A system has an *axis of symmetry* if the system is left unchanged by a rotation of arbitrary magnitude about the axis. It is not difficult to show that the mass center lies on the axis of symmetry.

Thus, for example, it is evident that

(i) The mass center of a solid sphere lies at its geometrical center, when the sphere is homogeneous or when the density depends only on the distance from the center.

(ii) The mass center of a solid homogeneous hemisphere lies on the radius which is perpendicular to its plane face.

(iii) The mass center of a plate in the form of an equilateral triangle (of

where  $G$  is a universal constant, called the constant of gravitation. The forces act along the line joining the particles, in accordance with the law of action and reaction stated in Sec. 1.4.

If we think of the particle of mass  $m_1$  as fixed and that of mass  $m_2$  as free to take up various positions, we recognize that the mass  $m_1$  produces a field of force. It is usual to take  $m_2 = 1$  for simplicity in discussing this field; then the magnitude of the force of attraction is  $Gm_1/r^2$ .

If we take coordinates with origin  $O$  at  $m_1$ , the direction cosines of the line drawn from  $O$  to any point  $A$  with coordinates  $x, y, z$  are  $x/r, y/r, z/r$ . Hence the components of force on unit mass at  $A$  are

$$(3.117) \quad X = -\frac{Gm_1x}{r^3}, \quad Y = -\frac{Gm_1y}{r^3}, \quad Z = -\frac{Gm_1z}{r^3},$$

the minus sign occurring since the force is directed from  $A$  toward  $O$ . Now

$$r^2 = x^2 + y^2 + z^2, \quad r \frac{\partial r}{\partial x} = x, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = -\frac{x}{r^3};$$

therefore (3.117) may be written

$$(3.118) \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z},$$

where

$$(3.119) \quad V = -\frac{Gm_1}{r}.$$

This is the potential energy (cf. 2.419) of a particle of unit mass in the gravitational field of a particle of mass  $m_1$ , or, briefly, the *potential* of the field. Thus *the force of attraction is the gradient of the potential, with sign reversed*. When a number of attracting particles are present, the resultant force of attraction is the vector sum of the individual forces of attraction. This resultant force is equal to the negative of the gradient of the total potential, i.e., the sum of the potentials due to the several particles. In calculating the force of attraction due to a system of particles (or a continuous distribution of matter), it is often convenient to find the potential first.

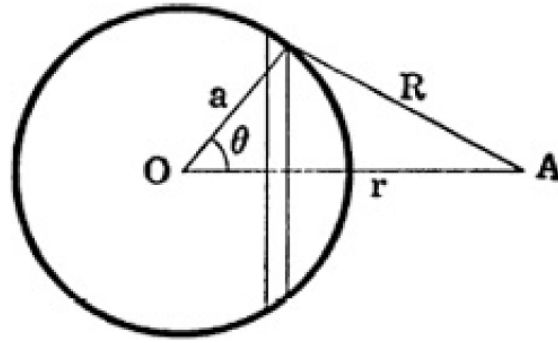


FIG. 29.—A spherical shell divided into thin rings for the calculation of the potential at  $A$ .

Let us consider a thin spherical shell of matter of radius  $a$  (Fig. 29). We wish to find the potential at an external point  $A$ , at a distance  $r$  from the center  $O$ .

Let us draw cones with  $O$  for vertex,  $OA$  for axis, and semivertical angles  $\theta, \theta + d\theta$ . These cut off from the shell a ring of area  $2\pi a^2 \sin \theta d\theta$ . The elements of this ring are all at the same distance ( $R$ ) from  $A$ , and so the potential due to the ring is

$$-2\pi G\sigma a^2 \sin \theta d\theta/R,$$

where  $\sigma$  is the mass per unit area of the shell. Expressing  $R$  in terms of  $a, r, \theta$  and integrating over the shell, we find for the potential

$$\begin{aligned} (3.120) \quad V &= - \int_0^\pi \frac{2\pi G\sigma a^2 \sin \theta d\theta}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} \\ &= - \frac{2\pi G\sigma a}{r} [\sqrt{(r+a)^2} - \sqrt{(r-a)^2}], \end{aligned}$$

the positive values of the square roots being understood. Since  $r > a$ , the last square root is  $r - a$ , and so

$$(3.121) \quad V = - \frac{4\pi G\sigma a^2}{r} = - \frac{GM}{r},$$

where  $M$  is the total mass of the shell.

Thus we have the following result: *The potential (and hence the force of attraction) of a thin spherical shell at any external point is the same as if the whole mass of the shell were concentrated at its center.*

If the point  $A$  lies inside the shell instead of outside, we proceed as before down to (3.120). But now  $a > r$ , and so the last square root is  $a - r$ . Hence

$$V = -4\pi G\sigma a,$$

a constant. Thus, inside a thin spherical shell the potential is constant, and the force of attraction is zero.

We can now discuss the gravitational field of the earth, supposing it to be composed of thin spherical shells, each of constant density. Each shell attracts as if its mass were concentrated at the center of the earth. Hence we have the following result: *At a point A, outside the earth, the force of attraction is directed toward the center of the earth and is of magnitude*

$$(3.122) \quad \frac{GM}{r^2},$$

where  $M$  is the mass of the earth and  $r$  the distance of  $A$  from the center of the earth.

In particular, if  $r$  is the radius of the earth, (3.122) gives the force of attraction at the earth's surface. The constant  $G$  is very small ( $6.67 \times 10^{-8}$  c.g.s. unit), and so gravitational forces are insignificant unless the masses involved are great. For this reason we usually neglect the mutual attractions of bodies on the earth's surface in comparison with the earth's attraction.

### Centers of gravity.

We consider now a body near the earth's surface, the body being small in comparison with the earth's radius. (We have in mind a piece of laboratory apparatus or even a large engineering structure, but not anything which would be of appreciable size on a map of the world.) Throughout this body the direction and magnitude of the earth's attraction are nearly constant. This leads us to the construction of the following model for the discussion of gravity near the earth's surface: *The earth's surface (or the ground) is represented by a plane (the horizontal plane). The earth's attraction on a particle of mass  $m$  is of magnitude  $mg$ , where  $g$  is a constant; it is directed vertically downward (i.e., perpendicular to and toward the ground).* The value of  $g$  is approximately  $32 \text{ ft. sec.}^{-2}$ , or  $980 \text{ cm. sec.}^{-2}$ .

We shall now show that there is just one point  $C$ , the *center of gravity* of a body, which satisfies the following conditions:

(i) *The potential energy of the body is equal to that of a single particle with mass equal to the total mass of the body, situated at  $C$ .*

(ii) *The whole system of forces due to gravity is plane-equipollent (with respect to any vertical plane) to a single vertical force through  $C$ .*

Let us take axes  $Oxyz$ ,  $Ox$  and  $Oz$  being horizontal and  $Oy$  vertical. Let us choose  $O$  as standard position for each of the particles forming the body. Then a particle of mass  $m_2$  at the point  $(x_1 \ y_1 \ z_2)$  has by (2.420) potential energy  $m_1gy_2$ , and so the whole potential energy is (for  $n$  particles)

$$(3.123) \quad V = g \sum_{i=1}^n m_i y_i.$$

Let the coordinates of  $C$  be  $x, y, z$ ; condition (i) is equivalent to

$$(3.124) \quad V = Mgy,$$

where  $M$  is the total mass of the body. Hence, comparing the two expressions for  $V$ , we have

$$y = \frac{\sum_{i=1}^n m_i y_i}{M}.$$

The condition of plane equipollence with respect to the plane  $z = 0$  demands that

$$Mgx = \sum_{i=1}^n m_i g x_i,$$

these being moments about  $Oz$ . Hence

$$x = \frac{\sum_{i=1}^n m_i x_i}{M}.$$

Similarly,

$$z = \frac{\sum_{i=1}^n m_i z_i}{M}.$$

Thus the center of gravity  $C$  exists, with coordinates

$$(3.125) \quad x = \frac{\sum_{i=1}^n m_i x_i}{M}, \quad y = \frac{\sum_{i=1}^n m_i y_i}{M}, \quad z = \frac{\sum_{i=1}^n m_i z_i}{M}.$$

We note, on referring to (3.103), that the center of gravity is in fact the same point as the mass center.

The force  $Mg$ , directed downward through the center of gravity, is called the *weight* of the body.

An accurate treatment of statics on the earth's surface is complicated by the



earth's rotation about its axis and its motion round the sun. However, the effects due to these causes are very small, and we may neglect them without making serious physical errors. In fact, we get satisfactory results by treating the earth as a Newtonian frame of reference. Likewise, another simplification introduced above (the assumption that the earth is flat, with a uniform gravitational field) does not cause serious physical errors. So, if we do not wish to obtain results of extremely high physical accuracy, we may use the model described above; this is, in fact, the procedure throughout the rest of the chapter.

The effects of the rotation of the earth are considered in Sec. 5.3 and also in Sec. 13.5. It will be shown that, as far as statics is concerned, this introduces no real complication; it merely modifies the value of  $g$ .

### 3.2. FRICTION

In Sec. 2.4 we introduced the concept of a smooth surface; the essential property is that, at a smooth contact, the reaction is normal to the surface. We shall now discuss the reaction at a rough contact and state the laws of friction.

#### Laws of static and kinetic friction.

Let  $A$  and  $B$  (Fig. 30) be two bodies in contact. Let  $\mathbf{R}$  be the reaction exerted by  $B$  on  $A$ .  $\mathbf{R}$  can be resolved in a unique manner into the forces  $\mathbf{N}$  and  $\mathbf{F}$ ,  $\mathbf{N}$  lying along the normal at the point of contact and  $\mathbf{F}$  lying in the plane of contact.  $\mathbf{N}$  is called the *normal reaction* and  $\mathbf{F}$  the *force of friction*. (At a smooth contact,  $\mathbf{F} = \mathbf{0}$ .)

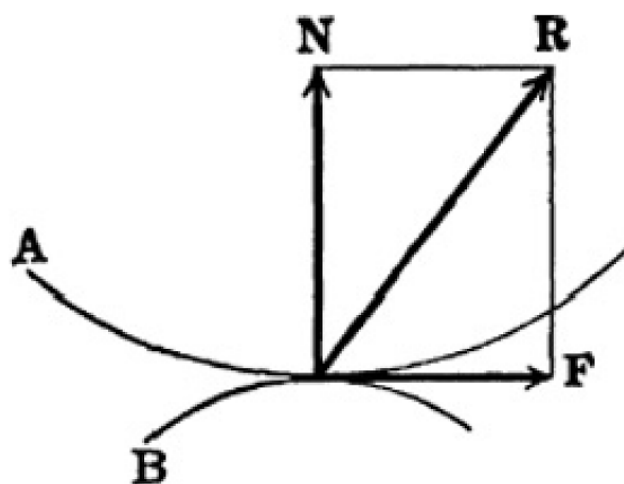


FIG. 30.—The reaction  $\mathbf{R}$  at a rough contact resolved into the normal reaction ( $\mathbf{N}$ ) and the force of friction ( $\mathbf{F}$ ).

On the basis of experiment, certain *laws of friction* are accepted. These are mathematical idealizations from the experimental results, and a high degree of accuracy in predictions based on these laws is not to be expected.

**LAW OF STATIC FRICTION.** *When two surfaces are in contact and no slipping*

overcome by treating cases of *limiting friction*, for which

$$(3.206) \quad \frac{F}{N} = \mu.$$

When this relation holds, the system is on the point of slipping.

### Some problems on friction.

*Example 1. A light ladder is supported on a rough floor and leans against a smooth wall. How far up the ladder can a man climb without slipping taking place?*

In Fig. 31,  $AB$  is the ladder and  $C$  is the man (replaced by a particle). Only three forces act on the ladder: (i) the weight of the man ( $W$ ); (ii) the reaction at the wall, this reaction being horizontal on account of the smoothness of the wall; (iii) the reaction of the ground. The lines of action of the first two meet at  $D$ . Hence the line of action of (iii) must pass through  $D$ , and hence the angle  $DBE$ , where  $BE$  is vertical, must not exceed the angle of friction  $\lambda$ . Thus the highest position that the man can reach may be found as follows: Draw a line through  $B$ , making an angle  $\lambda$  with  $BE$ ; let it cut the horizontal through  $A$  at  $D$ ; through  $D$ , draw a vertical; the point  $C$  where this line cuts the ladder is the required highest position. This method is called *descriptive* or *graphical*, because the result may be obtained by drawing to scale.

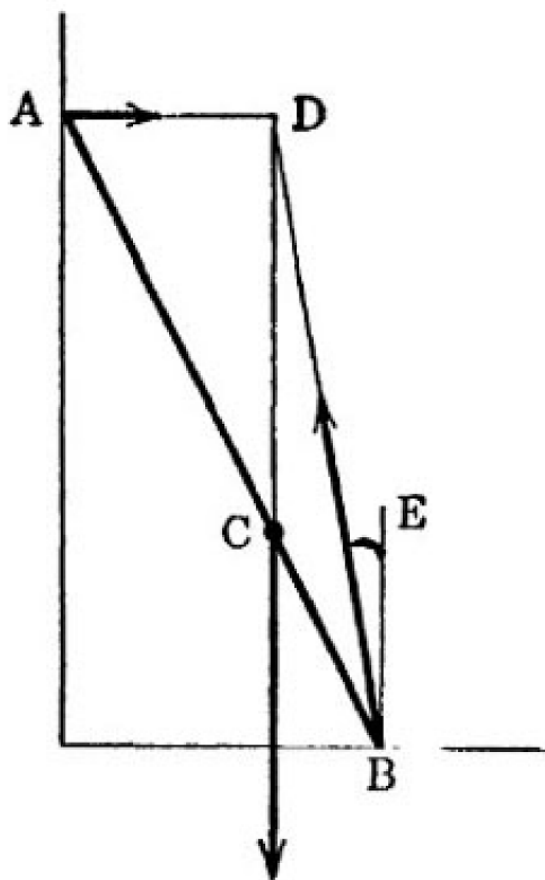


FIG. 31.—The ladder problem for a smooth wall (descriptive method).

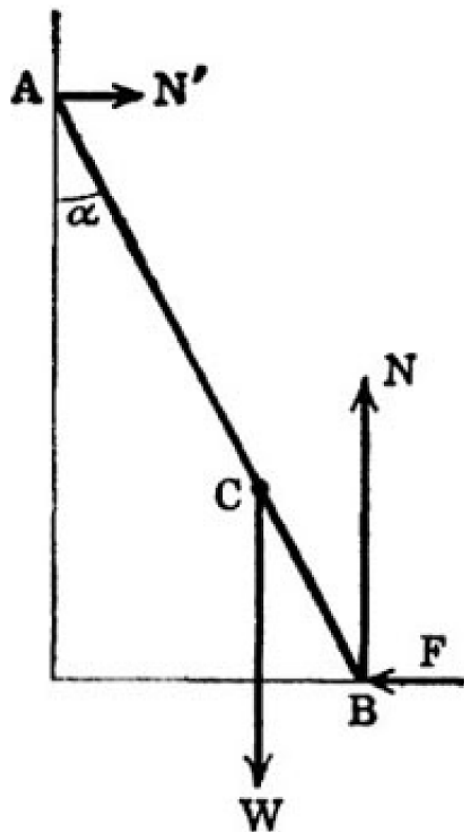


FIG. 32.—The ladder problem for a smooth wall (analytical method).

Let us now discuss the same problem analytically. Figure 32 shows the forces acting on the ladder. Let  $\alpha$  be the inclination of the ladder to the vertical. The total vertical component must vanish; thus

$$N - W = 0.$$

The total horizontal component must vanish; thus

$$N' - F = 0.$$

The total moment about  $B$  must vanish; thus

$$W \cdot BC \sin \alpha - N' \cdot AB \cos \alpha = 0.$$

Hence

$$\begin{aligned} F &= N' = W \frac{BC}{AB} \tan \alpha, \\ N &= W, \\ \frac{F}{N} &= \frac{BC}{AB} \tan \alpha. \end{aligned}$$

Thus, by (3.202),

$$\frac{BC}{AB} \tan \alpha \leq \mu.$$

The highest point  $C$  attainable is given by

$$(3.207) \quad BC = AB\mu \cot \alpha.$$

The analytical method appears more complicated than the descriptive, but it has the advantage of being more systematic. Moreover, since the three conditions of equilibrium give all possible information, the solution of the problem is reduced to algebra as soon as they are written down.

It might be thought that in drawing the arrow for the force of friction to the left in Fig. 32, we were anticipating the result. This is not actually the case. When we draw an arrow in connection with a component of a force, we are simply indicating the sense in which this component is considered positive. Had we drawn the arrow to the right in Fig. 32, we should have obtained equations as above, but with the sign of  $F$  reversed. The final physical result would have been the same.

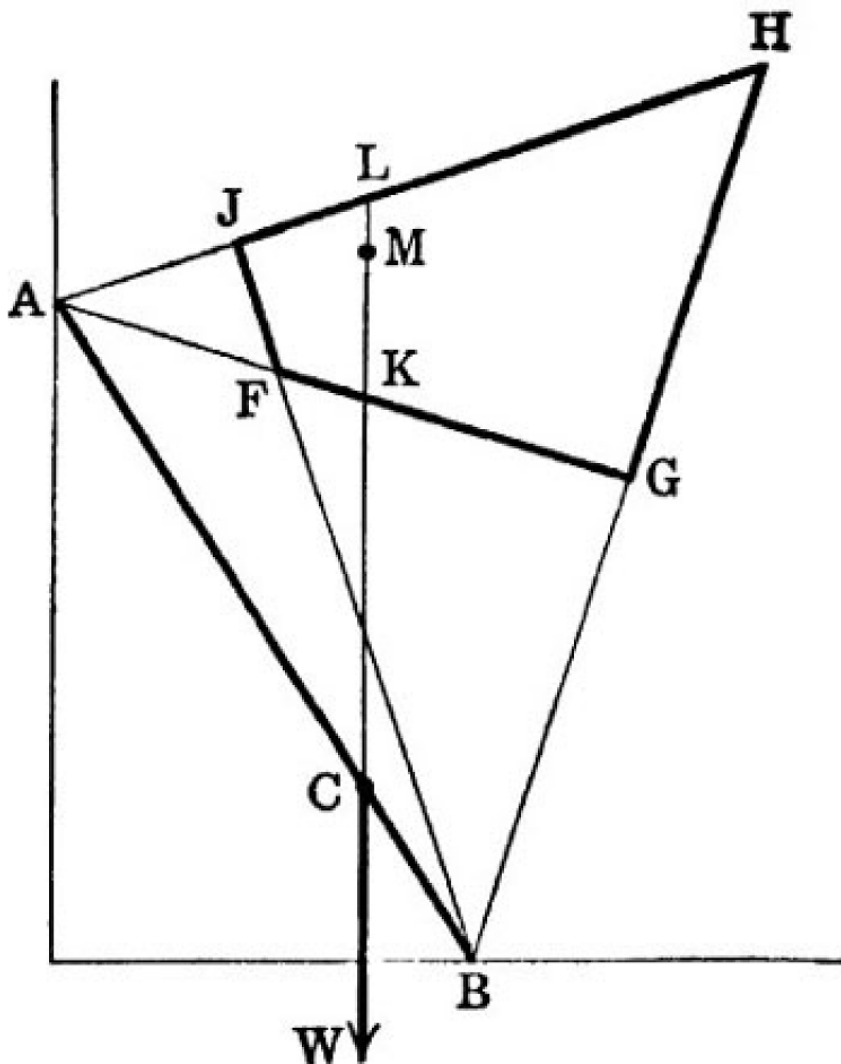


FIG. 33.—The ladder problem for a rough wall (descriptive method).

However, since positive quantities are easier to think of than negative quantities, it is advisable whenever possible to draw the arrows in the senses in which the forces really act. Thus, in the case of  $N$ , we draw the arrow upward. As for friction, it is generally found that the force of friction acts in the direction opposed to the motion which would take place in its absence. That is why the arrow for  $F$  in Fig. 32 was drawn to the left.

*Example 2. The preceding problem modified by supposing both wall and floor to be rough, with the same coefficient of friction  $\mu$ .*

Consider the cones of friction at  $A$  and  $B$ . They will cut the plane of the paper in four lines as shown in Fig. 33, these four lines giving the quadrilateral  $FGHJ$ . Draw the vertical through  $C$ , the position of the man, and let this vertical cut the sides of the quadrilateral at  $K, L$ . Let  $M$  be any point on the segment  $KL$ . Now the weight  $W$  may be resolved into forces along  $MA, MB$ , and hence  $W$  can be balanced by forces along  $AM, BM$ . Since these lines lie inside the cones of friction, the law of friction is satisfied. We have here a case of statical indeterminacy (cf. Sec. 2.5): provided that the vertical through  $C$  cuts the quadrilateral  $FGHJ$ , the ladder will be in equilibrium, but we cannot tell precisely what the reactions of the wall and floor will be.

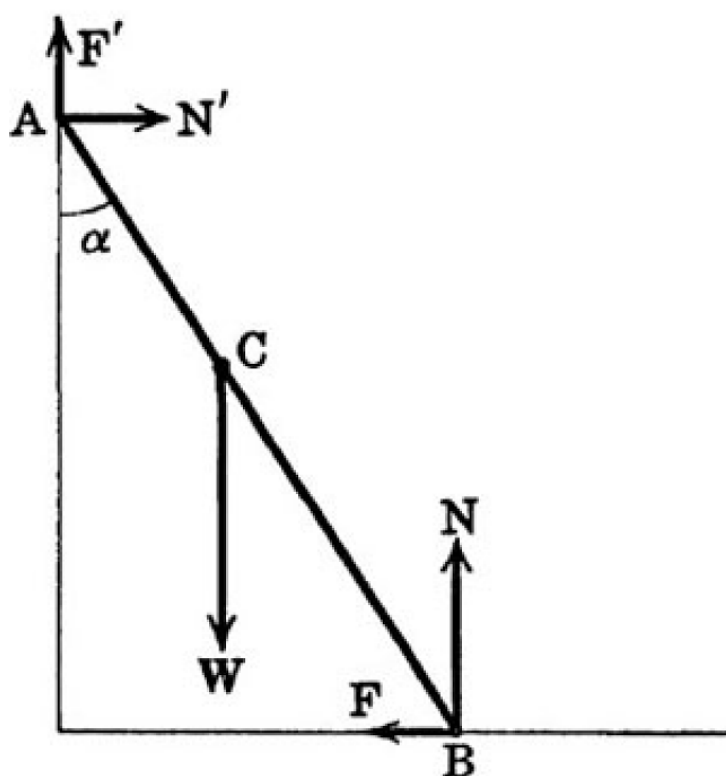


FIG. 34.—The ladder problem for a rough wall (analytical method).

Now let us ask: How far can the man go up the ladder before slipping takes place? Obviously, he can climb until the vertical through his position passes through the point  $J$ . When he passes that position, it will no longer be possible to



find reactions satisfying the conditions of equilibrium and the law of friction.

The question may also be treated analytically. Consider the man slowly climbing the ladder. If the ladder slips at all, just at the point of slipping the reactions at *both* contacts must correspond to limiting friction. Thus, at the point of slipping, the force system is as shown in Fig. 34, with  $F = \mu N$ ,  $F' = \mu N'$ . Taking vertical and horizontal components and moments about  $B$ , we have the three equations

$$\begin{aligned}\mu N' + N - W &= 0, \\ N' - \mu N &= 0, \\ W \cdot BC \sin \alpha - \mu N' \cdot AB \sin \alpha - N' \cdot AB \cos \alpha &= 0.\end{aligned}$$

These three equations determine  $N$ ,  $N'$ ,  $BC$ : we find

$$(3.208) \quad N = \frac{W}{1 + \mu^2}, \quad N' = \frac{\mu W}{1 + \mu^2}, \quad \frac{BC}{AB} = \frac{\mu}{1 + \mu^2} (\mu + \cot \alpha).$$

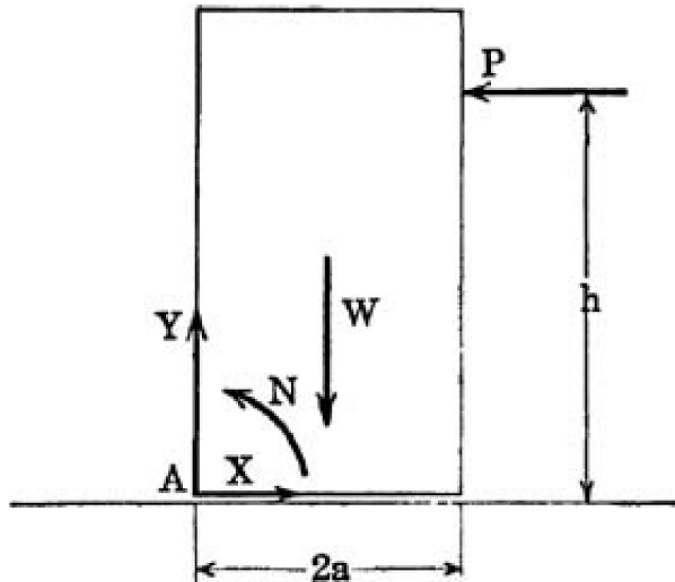


FIG. 35.—A heavy block pushed by a horizontal force  $P$ .

*Example 3. A block rests on a rough horizontal floor and is pushed by a gradually increasing horizontal force. Will the block slide, or will it topple over an edge?*

Let the thickness of the block be  $2a$ , its weight  $W$ , and the coefficient of static friction  $\mu$ . Let the horizontal force  $P$  be applied at a height  $h$  above the floor. The first question is: Given  $W$  and  $P$  as shown in Fig. 35, can there be a system of reactions exerted by the ground, satisfying simultaneously the law of friction and the conditions of equilibrium for the block? Any such system of reactions will be plane-equipollent to forces  $X$ ,  $Y$  at  $A$  as shown, together with a couple  $N$ . If equilibrium exists, it is clear that the following conditions are demanded by the law of friction and the fact that the floor cannot pull the block downward:

$$(3.209) \quad X \leq \mu Y, \quad Y \geq 0, \quad N \geq 0.$$

Taking horizontal and vertical components and moments about  $A$ , we have

$$X = P, Y = W, N = aW - hP,$$

and so (3.209) give

$$(3.210) \quad P \leq \mu W, \quad P \leq \frac{aW}{h}.$$

Starting with a small value of  $P$ , these inequalities are both satisfied; but as  $P$  is increased, one or other will be violated, and then equilibrium will cease. If

$$(3.211) \quad \mu < \frac{a}{h},$$

the first inequality of (3.210) will be broken first. At the instant when  $P = \mu W$ , we have

$$X = \mu Y, N > 0.$$

This is a state of limiting friction; and so, if (3.211) holds, equilibrium of the block will be broken by sliding along the plane. On the other hand, if

$$(3.212) \quad \mu > \frac{a}{h},$$

then the second inequality of (3.210) will be violated first. At the instant when  $P = aW/h$ , we have

$$X < \mu Y, Y > 0, N = 0.$$

The friction is not limiting, and so slipping cannot take place. But any further increase in  $P$  will cause violation of the last inequality of (3.209). Hence we conclude that, if (3.212) holds, equilibrium will be broken by the block turning over the edge  $A$ .

The result is in agreement with common experience: the smaller we make  $h$ , the more likely is sliding to occur.

### 3.3. THIN BEAMS

#### Tension, shearing force, and bending moment.

Let us consider a straight beam of uniform section (Fig. 36) and a plane  $P$  parallel to its length.  $P$  may be regarded as the

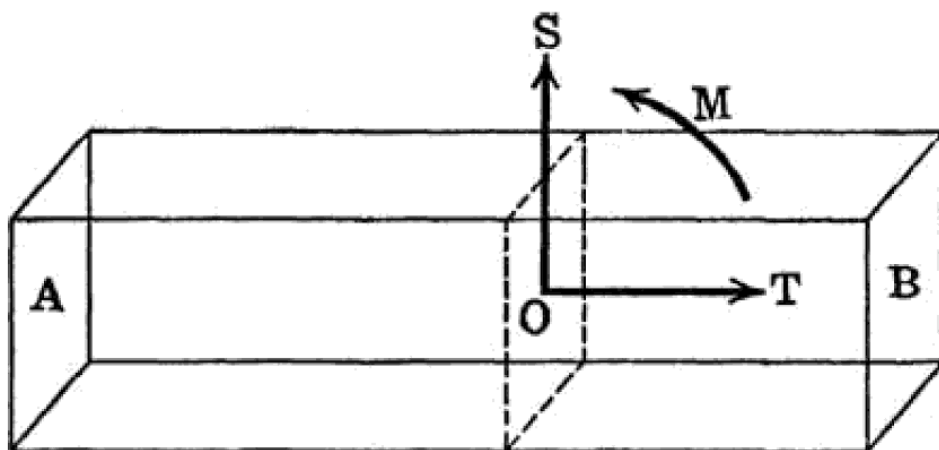


FIG. 36.—Reactions across a section of a beam.

plane of the paper. External forces, parallel to  $P$ , act on the beam. (These forces are not shown. They may consist of the weight of the beam or loads placed on it.) Let a cross section be drawn through a point  $O$ , perpendicular to the length of the beam. Let us take as our “system” the portion of the beam extending from the end  $A$  up to this section. The external forces acting on this system will consist of

- (i) the external forces already mentioned, acting on this portion of the beam,
- (ii) the reactions exerted across the section by the particles in the portion of the beam extending from the section to the end



FIG. 37.—A thin beam.

$B$ . These reactions are *internal* forces as far as the whole beam is concerned, but they are *external* forces for the system at present under consideration.

Let us take  $P$  as the fundamental plane. The reactions across the section are plane-equipollent to a force acting at  $O$ , together with a couple  $M$ . The force may be resolved into components  $T$ ,  $S$  along the beam and perpendicular to its length, respectively. We define the following terms:

$T$  = tension,

$S$  = shearing force,

$M$  = bending moment.

We shall confine our attention to *thin beams*. The thin beam is a mathematical idealization, in which the cross section is reduced to a point and the beam to a straight line.

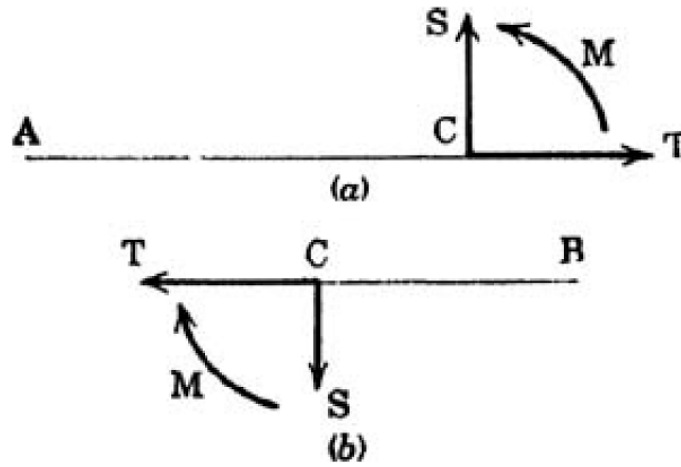


FIG. 38.—(a) Reactions exerted on  $AC$  by  $CB$ . (b) Reactions exerted on  $CB$  by  $AC$ .

Figure 37 shows a thin beam  $AB$ ;  $C$  is any point of it. To draw the reactions on  $AC$  across the section at  $C$  without confusion, we delete the line  $CB$  as in Fig. 38a. Figure 38b shows the reactions on  $CB$ ; these have the same magnitudes as, but opposite senses to, those shown in Fig. 38a, on account of the law of action and reaction.

Let us take an origin on the beam, the  $x$ -axis along the beam and the  $y$ -axis perpendicular to it. Consider a small length of the beam extending from  $x$  to  $x + dx$  (Fig. 39). Let  $T$ ,  $S$ ,  $M$  be the values of tension, shearing force, and bending moment at  $x$ , and  $T + dT$ ,  $S + dS$ ,  $M + dM$  the values at  $x + dx$ . To allow for gravity or other continuous external loading, we shall add a force with components  $X dx$ ,  $Y dx$  (not shown in Fig. 39) acting at the middle point of the portion  $x$ ,  $x + dx$ . By taking components and moments about the point  $x$  and neglecting

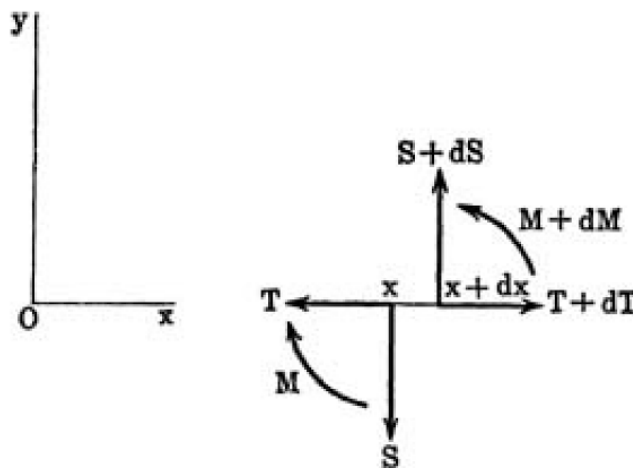


FIG. 39.—Reactions on the ends of a small element of a beam.

infinitesimals of the second order, we have, as conditions of equilibrium for the small length of the beam,

$$dT + X dx = 0, dS + Y dx = 0, dM + S dx = 0.$$

Thus

$$(3.301) \quad \frac{dT}{dx} = -X, \quad \frac{dS}{dx} = -Y, \quad \frac{dM}{dx} = -S.$$

These are the general differential equations for the equilibrium of thin beams. But in statically determinate cases we can obtain all required information regarding internal reactions without using these equations, or rather by using them in integrated form.

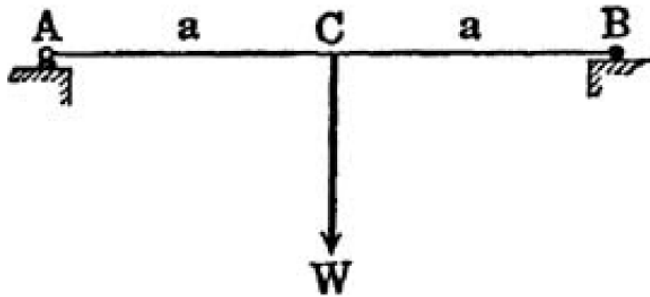


FIG. 40.—A light beam loaded at its middle point.

### Statically determinate problems.

We shall illustrate the method by the solution of a problem. A light beam  $AB$  of length  $2a$  is hinged at  $A$  and supported on a smooth horizontal plane at  $B$  (Fig. 40). A load  $W$  is placed at the middle point  $C$ . Find the bending moment and shearing force along the beam.

First, by application of the conditions of equilibrium (2.306) to the whole beam, we find the reactions on the beam at  $A$  and  $B$ . These are each of magnitude  $\frac{1}{2}W$ , directed upward. Let us take our origin at  $A$  and the  $x$ -axis along the beam. Consider the

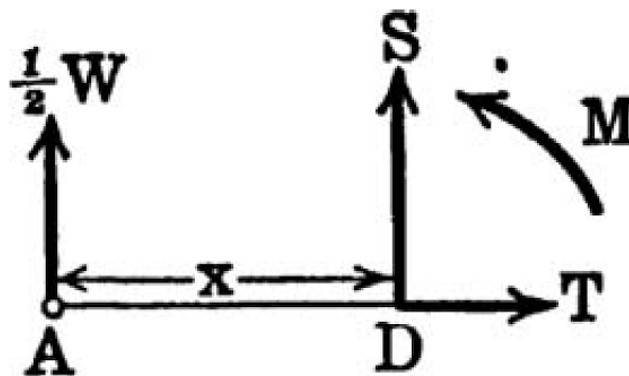


FIG. 41.—External forces on a portion of the beam shown in Fig. 40 ( $AD < AC$ ).

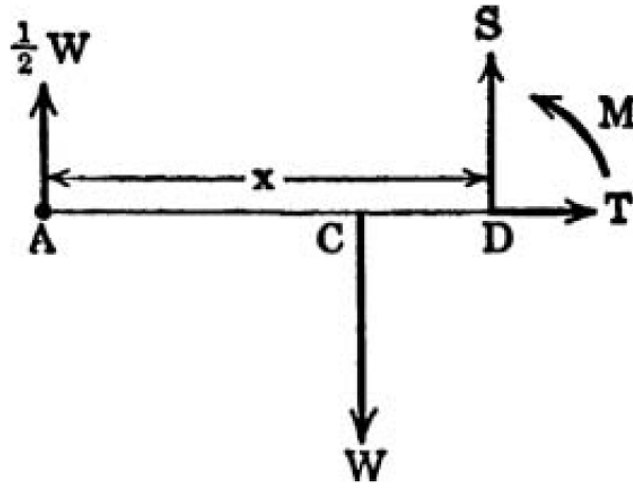


FIG. 42.—External forces on a portion of the beam shown in Fig. 40 ( $AD > AC$ ).

portion of the beam  $AD$ , where  $D$  lies in  $AC$ ; let  $AD = x$  (Fig. 41). From the equilibrium of  $AD$ , we have

$$(3.302) \quad \begin{cases} T = 0, & S = -\frac{1}{2}W, \\ M = -xS = \frac{1}{2}xW, & (x < a). \end{cases}$$

These give the shearing force and bending moment for any point in  $AC$ ; there is no tension. Since  $S$  is negative, the shearing force actually acts in the *downward* direction.

Now take  $D$  in  $CB$  (Fig. 42). Instead of (3.302), we have

$$(3.303) \quad \begin{cases} T = 0, & S = W - \frac{1}{2}W = \frac{1}{2}W, \\ M = aW - xS = (a - \frac{1}{2}x)W, & (x > a). \end{cases}$$

The shearing force is now positive. We note that the last of (3.301) is satisfied by (3.302) and (3.303).

The graphs of  $S$  and  $M$  along the beam are shown in Fig. 43.

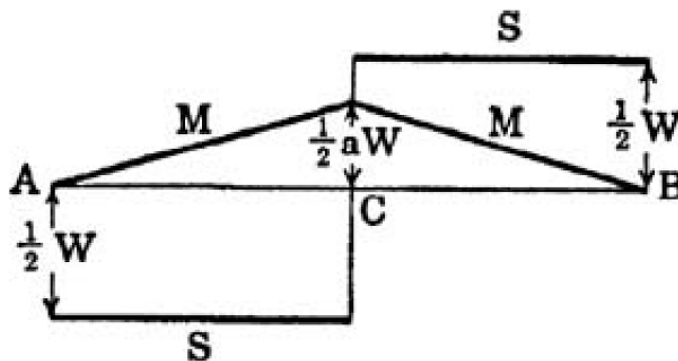


FIG. 43.—Graphs of shearing force ( $S$ ) and bending moment ( $M$ ) along the beam shown in Fig. 40.

## The Euler-Bernoulli theory of thin elastic beams.



If a Straight beam rests On three supports, the problem of finding the reactions due to the supports is statically indeterminate (cf. Sec. 2.5), and we cannot find the shearing force and bending moment by elementary statical principles. But this indeterminacy disappears when we take into consideration the elasticity of the beam. Although straight initially, an elastic beam will stretch and bend under the influence of forces. We suppose the stretching and bending to be very small and accept the law of Hooke for stretching and the law of Euler and Bernoulli for bending.\*

HOOKE'S LAW. When a beam is slightly stretched,

$$(3.304) \quad T = k'e,$$

where  $e$  is the extension (increase in length per unit length) and  $k'$  a constant for the beam. (Actually  $k' = EA$ , where  $E$  is Young's modulus for the material and  $A$  the area of the cross section.)

THE EULER-BERNOULLI LAW. When a beam is slightly bent, the bending moment is connected with the curvature by the relation

$$(3.305) \quad M = \frac{k}{\rho},$$

where  $\rho$  is the radius of curvature and  $k$  a constant for the beam. (Actually  $k = EI$ , where  $E$  is Young's modulus and  $I$  the “moment of inertia” of the cross section about an axis through its mean center perpendicular to the plane of the couple  $M$ .)

When the beam is approximately straight and the axes as in Fig. 39,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \quad \text{approximately,}$$

and (3.305) may be written

$$(3.306) \quad M = k \frac{d^2y}{dx^2}.$$

Let us refer to Fig. 39 and to the equations (3.301). We shall suppose that the beam is subject to a force  $w$  per unit length in the negative sense of the  $y$ -axis, due either to its own weight or to a load placed on it. Then  $X = 0$ ,  $Y = -w$ , and (3.301) read

$$(3.307) \quad \frac{dT}{dx} = 0, \quad \frac{dS}{dx} = w, \quad \frac{dM}{dx} = -S.$$

We see that the tension  $T$  is constant. Elimination of  $M$  and  $S$  from (3.306) and (3.307) gives

$$(3.308) \quad k \frac{d^4 y}{dx^4} = -w.$$

This is the fundamental differential equation in the theory of thin elastic beams. If it is solved, the bending moment and shearing force are given by

$$(3.309) \quad M = k \frac{d^2 y}{dx^2}, \quad S = -\frac{dM}{dx}.$$

It must be realized that the differential equation (3.308) holds only between isolated loads or supports. To deal with these a special treatment is necessary. Figure 44 shows an element  $\Delta x$  of a thin beam with an isolated load  $W$  suspended from its middle point  $P$ . (The case of a support is covered by making  $W$  negative.)

The element is in equilibrium under four forces and two couples: the continuous load on  $\Delta x$  (not shown), the isolated load  $W$ , the shearing force  $S + \Delta S$ , and the bending moment  $M + \Delta M$  on the right, and the shearing force  $S$  and the bending moment  $M$  on the left, positive senses being as indicated.

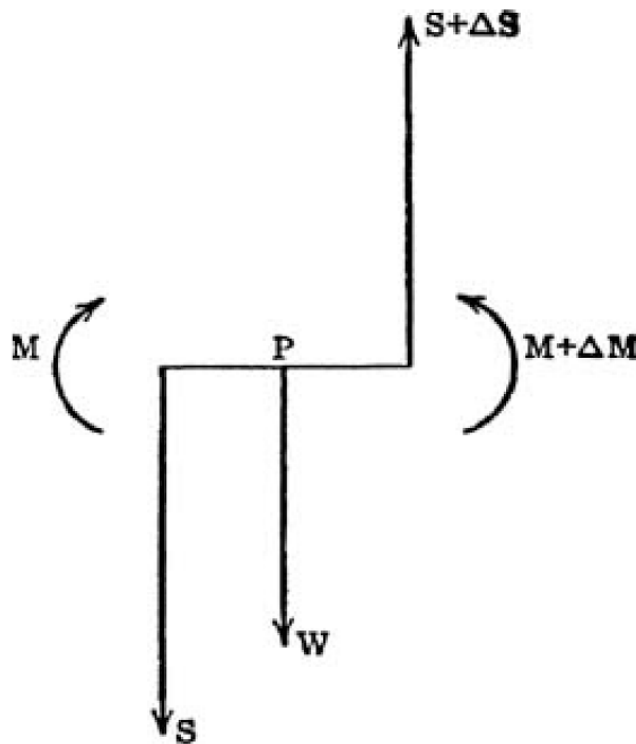


FIG. 44.—Element of beam containing isolated load.

If  $\Delta x$  tends to zero, the continuous load tends to zero and so does the moment of this load about  $P$ . Hence the conditions of equilibrium give, in the limit,  $\Delta S = W$ , and (taking moments about  $P$ )  $\Delta M = 0$ .

This means that the bending moment  $M$  is continuous across an isolated load or support, but the shearing force  $S$  changes abruptly. In terms of  $y$  and its derivatives (since the beam is not broken at the isolated load or support) we have continuity in  $y$ ,  $dy/dx$ ,  $d^2y/dx^2$ , but discontinuity in  $d^3y/dx^3$ .

*Example.* A uniform heavy beam  $OP$  of length  $2a$  and weight  $W$  is hinged at  $O$  and rests on two smooth supports, one at  $P$  and the other at its middle point  $Q$ . Find the reactions on the supports, if  $O, P, Q$  are all at the same height.

We shall take the origin of coordinates at  $O$ , the  $x$ -axis horizontal, and the  $y$ -axis directed vertically upward. Integration of (3.308) along  $OQ$  gives

$$(3.310) \quad ky = -\frac{1}{24}wx^4 + Ax^3 + Bx, \quad (OQ)$$

where  $A, B$  are constants of integration; two other constants of integration have been put equal to zero on account of the vanishing of  $y$  and  $d^2y/dx^2$  at  $O$  (There can be no bending moment at a hinge or free end) Similarly, we have along  $QP$

$$(3.311) \quad ky = -\frac{1}{24}w(x-2a)^4 + A'(x-2a)^3 + B'(x-2a), \quad (QP)$$

where  $A', B'$  are constants of integration. In these two equations, we have four unknown constants; they are to be found from the conditions that  $y = 0$  at  $Q$ , while  $dy/dx$  and  $d^2y/dx^2$  are continuous there. Thus, we have the four equations

$$\begin{aligned} Aa^3 + Ba - \frac{1}{24}wa^4 &= 0, \\ A'a^3 + B'a + \frac{1}{24}wa^4 &= 0, \\ 3Aa^2 + B - \frac{1}{6}wa^3 &= 3A'a^2 + B' + \frac{1}{6}wa^3, \\ 6Aa - \frac{1}{2}wa^2 &= -6A'a - \frac{1}{2}wa^2. \end{aligned}$$

We find

$$A = -A' = \frac{1}{16}wa, \quad B = -B' = -\frac{1}{48}wa^3,$$

and substitution in (3.310) and (3.311) gives the equations of the two portions of the beam

$$(3.312) \quad \begin{cases} ky = -\frac{1}{24}wx^4 + \frac{1}{16}wax^3 - \frac{1}{48}wa^3x, & (OQ), \\ ky = -\frac{1}{24}w(2a-x)^4 + \frac{1}{16}wa(2a-x)^3 - \frac{1}{48}wa^3(2a-x), & (QP) \end{cases}$$

In  $OQ$  the bending moment is

$$M = k \frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{3}{8}wax.$$

Its maximum value occurs at  $x = \frac{3}{8}a$ . The portion  $OQ$  is a system in equilibrium; hence, taking moments about  $Q$ , we have for the reaction  $R_O$  at  $O$

$$R_Oa = M_Q + \frac{1}{2}wa^2 = \frac{3}{8}wa^2 = \frac{3}{16}Wa.$$

When one reaction has been found, the others follow from the usual statical methods. Hence

$$(3.313) \quad R_O = \frac{3}{16}W, \quad R_Q = \frac{11}{16}W, \quad R_P = \frac{3}{16}W.$$

### 3.4. FLEXIBLE CABLES

A flexible cable differs from a stiff rod in the ease with which it can be bent into a curve. The bending moment per unit curvature is much less for the cable. In mechanics, we idealize this property and understand by a flexible cable a material curve such that there can be no bending moment across any section. By considering the equilibrium of a small portion of the cable, it is easily seen that the shearing force must also vanish. Hence the only surviving component of the reaction across a section of a flexible cable is a tension  $T$ , which acts along the tangent to the curve in which the cable lies.

We use the word “cable ” exclusively, but it is to be understood that the practical applications cover chains, ropes, strings, and threads. The theoretical predictions will agree well with the results of experiments conducted on cables in which the bending moments are small.

### General formulas for all flexible cables hanging freely.

Let us consider a flexible cable hanging under the influence of its own weight, and perhaps additional continuous vertical loads attached to it. For the present, we shall not make any special assumptions regarding the nature of the cable or the load.

We pass over the trivial case in which the cable hangs from one end in a vertical line. When suspended from two points, it hangs in a vertical plane. Let  $Oxy$  be axes in this plane,  $Ox$  being horizontal and  $Oy$  directed vertically upward (Fig. 45). Let  $A$  be a point on the cable with coordinates  $(x, y)$ , and  $B$  an adjacent point with coordinates  $(x + dx, y + dy)$ . Let  $ds$  be the infinitesimal length of  $AB$ , and let  $w ds$  be the total load on  $AB$ , including the weight of the cable. The portion  $AB$  is a system in equilibrium under the action of the tensions at its ends and the load. Let  $\theta$  be the inclination of the tangent at  $A$  to the horizontal. Then  $dx/ds = \cos \theta$ ,  $dy/ds = \sin \theta$ ; and so, taking horizontal and vertical components, we have

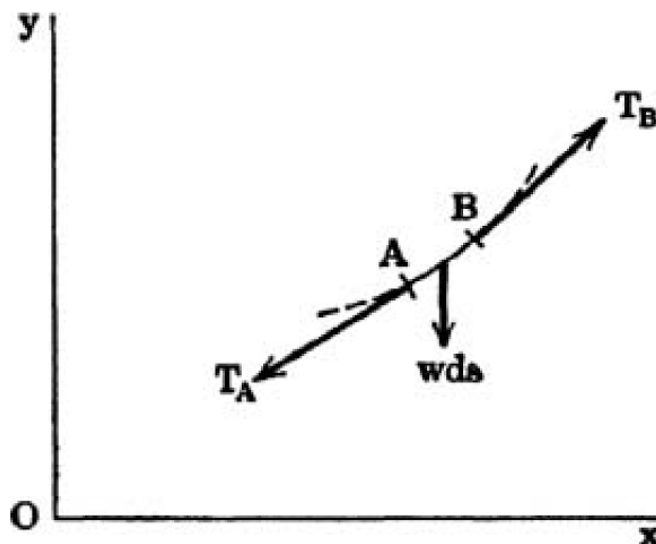


FIG. 45.—Forces acting on an element of a hanging cable.

$$\begin{aligned} \left(T \frac{dx}{ds}\right)_B - \left(T \frac{dx}{ds}\right)_A &= 0, \\ \left(T \frac{dy}{ds}\right)_B - \left(T \frac{dy}{ds}\right)_A - w ds &= 0. \end{aligned}$$

By the first of these equations, *the horizontal component of the tension is constant*. The second equation may be written

$$(3.401) \quad \frac{d}{ds} \left( T \frac{dy}{ds} \right) = w.$$

If  $H$  is the constant horizontal component of tension, we have

$$(3.402) \quad T \frac{dx}{ds} = H;$$

substitution in (3.401) gives

$$(3.403) \quad \frac{d}{ds} \left( \frac{dy}{dx} \right) = \frac{w}{H}.$$

This is a differential equation satisfied by the curve in which the cable hangs. When this equation has been solved, the tension may be found from (3.402).

### The suspension bridge.

Let us now suppose that a weightless cable supports a load uniformly distributed on a horizontal line; for the load on a horizontal length  $dx$ , we write  $w_0 dx$ . This approximates to the condition of a cable of a suspension bridge (Fig. 46), the load consisting of the roadway  $AB$ , of weight  $w_0$  per unit length.

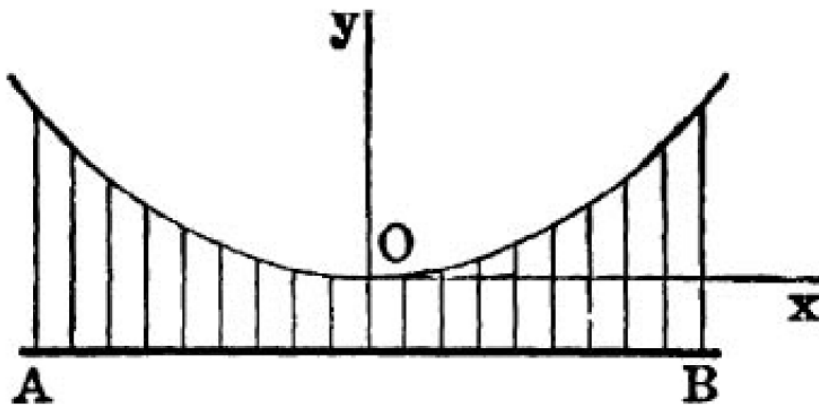


FIG. 46.—Suspension bridge.

With the notation used above, we have  $w \, ds = w_0 \, dx$ , and so

$$w = w_0 \frac{dx}{ds};$$

thus (3.403) reads

$$\frac{d}{ds} \left( \frac{dy}{dx} \right) = \frac{w_0}{H} \frac{dx}{ds},$$

or

$$\frac{d^2y}{dx^2} = \frac{w_0}{H}.$$

If the origin is chosen at the lowest point of the cable, so that  $y = dy/dx = 0$  when  $x = 0$ , we obtain as the equation of the cable

$$(3.404) \quad y = \frac{1}{2} \frac{w_0 x^2}{H}.$$

This is a *parabola*. The tension in the cable is given by (3.402). Since

$$(3.405) \quad ds^2 = dx^2 + dy^2, \quad \frac{ds}{dx} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2},$$

we have

$$(3.406) \quad T = H \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = \sqrt{H^2 + w_0^2 x^2}.$$

### The common catenary.

We shall now consider a uniform cable hanging freely under its own weight,  $w$  per unit length. The fundamental equation is (3.403), in which  $w$  is now a constant. We write it in the form

$$\frac{d^2y}{dx^2} = \frac{w}{H} \frac{ds}{dx},$$

or, by (3.405),



$$(3.419) \quad \frac{\sinh(a/c)}{a/c} = \frac{l}{a}.$$

If tables of  $(\sinh X)/X$  are available, the numerical value of  $a/c$  may be obtained at once.\* The solution of the problem is given by (3.417) on inserting the value for  $c$ , found from (3.419).

If the ratio  $l/a$  is nearly unity, i.e., if the cable is only a little longer than the span, the solution of (3.419) for  $a/c$  is small, because

$$\lim_{X \rightarrow 0} \frac{\sinh X}{X} = 1.$$

In fact, the parameter  $c$  is large. Then we can obtain an approximate solution of (3.419) without recourse to numerical tables. Retaining only the first two terms of the expansion for  $\sinh a/c$ , we have

$$(3.420) \quad 1 + \frac{1}{6} \left( \frac{a}{c} \right)^2 = \frac{l}{a}, \quad \frac{a}{c} = \sqrt{\frac{6(l-a)}{a}},$$

$$c = \sqrt{\frac{a^3}{6(l-a)}}.$$

Since  $c$  is large,  $T_{\max}$  as given by (3.417) is large; it is approximately equal to  $H$ , where

$$(3.421) \quad H = wc = wa \sqrt{\frac{a}{6(l-a)}}.$$

The second problem arises when the distance between two points  $A$  and  $B$  at the same height is measured by a measuring tape which sags under its own weight. With the notation of Fig. 48, we are given  $h, l$ ; we wish to find  $a$ .

Applying the second of (3.415) at the point  $B$ , we have

$$(3.422) \quad c = \frac{l^2 - h^2}{2h}.$$

The answer to the problem is given by (3.418). This is a quadratic equation for  $e^{a/c}$ , and the positive root gives†

$$(3.423) \quad a = c \log \left( \frac{l}{c} + \sqrt{1 + \frac{l^2}{c^2}} \right) = \frac{l^2 - h^2}{2h} \log \frac{l+h}{l-h}.$$

If the ratio  $h/l$  is small, we have approximately

$$(3.424) \quad \log \frac{l+h}{l-h} = 2 \left( \frac{h}{l} + \frac{1}{3} \frac{h^3}{l^3} \right),$$

and hence

$$(3.425) \quad a = l \left( 1 - \frac{2}{3} \frac{h^2}{l^2} \right),$$

with an error of the order of  $(h/l)^4$ .

### Cables in contact with smooth curves.

So far the cables considered have been unconstrained. Let us now consider the case of a cable lying against a smooth surface, or, as we may say in two-dimensional language, against a smooth curve. Gravity will be neglected.

Figure 49 shows a small portion  $AB$  of a cable lying in equilibrium in contact with a smooth curve. Let  $O$  be any assigned point on the cable and  $s$  the length of the cable between  $O$  and  $A$ . Let the length  $AB$  be  $ds$ , and let the inclinations to some fixed direction of the tangents to the cable at  $A$  and  $B$  be  $\theta$  and  $\theta + d\theta$ . The element  $AB$  is in equilibrium under three forces, namely, the tension  $T$  at  $A$ , the tension  $T + dT$  at  $B$ , and a normal reaction due to the curve. This last may be written  $N ds$  and may be supposed to act along the normal at  $A$ . Resolving forces along the tangent and normal at  $A$ , we obtain from the conditions of equilibrium

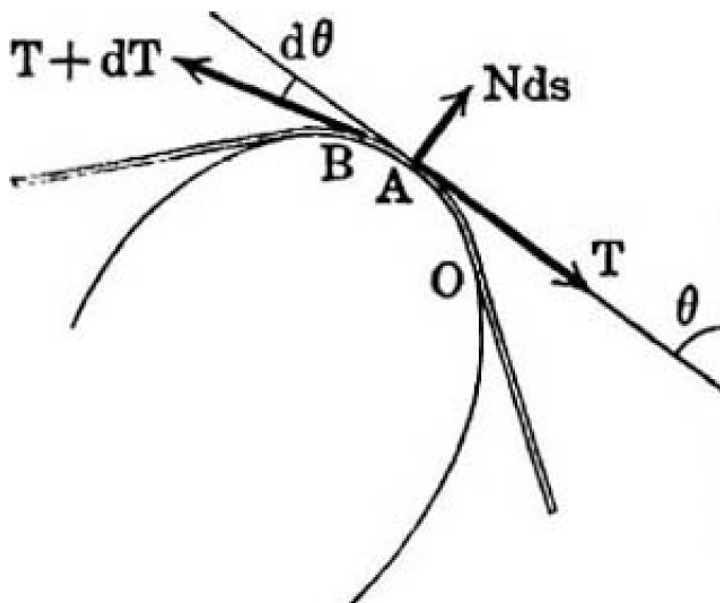


FIG. 49.—Forces on an element of light cable in contact with a smooth curve.

$$(3.426) \quad dT = 0, \quad N ds = T d\theta.$$

Hence, *the tension is constant along a light cable in contact with a smooth curve.* Also, since  $ds/d\theta = \rho$ , the radius of curvature, we have

$$(3.427) \quad N = \frac{T}{\rho}.$$

An example of the significance of this last formula occurs in tying up a parcel: the sharper the edge of the parcel, the smaller  $\rho$  and hence the greater the tendency of the string to bite into the parcel.

### Cables in contact with rough curves.

Let us now suppose that the curve shown in Fig. 49 is rough and that the cable is just on the point of slipping in the direction  $AB$ . In addition to the forces already considered, there is now a force of friction  $F ds$  on the element, acting along the tangent at  $A$  and opposing motion. The conditions of equilibrium are now

$$(3.428) \quad dT = F ds, \quad N ds = T d\theta.$$

But  $F = \mu N$ , where  $\mu$  is the coefficient of friction. Hence

$$(3.429) \quad \frac{dT}{ds} = \mu N, \quad N = T \frac{d\theta}{ds},$$

and so

$$(3.430) \quad \frac{dT}{d\theta} = \mu T.$$

Integration gives

$$(3.431) \quad T = T_0 e^{\mu\theta},$$

where  $T_0$  is a constant of integration.

The rapid increase of the exponential with increasing  $\theta$  is of great practical importance. As a numerical example, consider a rope wrapped twice around a post, for which the coefficient of friction is  $\frac{1}{2}$ . Then

$$T = T_0 e^{4\mu\pi} = T_0 e^{2\pi},$$

where  $T_0$ ,  $T$  are the tensions in the rope where it meets and leaves the post, slipping being about to occur in the direction of  $T$ . We have

$$\frac{T_0}{T} = e^{-2\pi} = 0.0019.$$

If  $T = 2000$  lb.,  $T_0 = 3.8$  lb. Thus, a load of one ton can be sustained by application of a force of less than 4 lb.; and, of course, a much greater load might be sustained if the rope were wrapped more often round the post. This principle is used in holding ships by ropes passed round mooring posts and in hoists in which a rope is passed round a revolving drum, the end being held in the hand.

### 3.5. FRAMES

#### Just-rigid frames.

Figure 50 shows a simple frame or truss, as used in bridges. It consists of steel girders riveted together at the joints. For mathematical discussion we simplify the system as follows: (i) the girders are treated as light rigid bars, (ii) the joints are

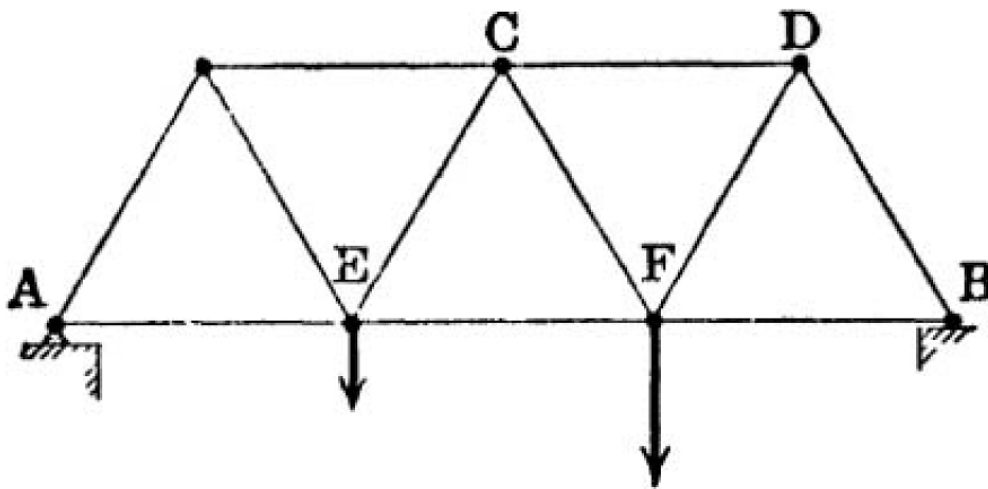


FIG. 50.—A just-rigid frame with loads applied at  $E$  and  $F$ .

supposed to be smoothly working hinges, each bar being capable of rotation about the joints on it without any resisting couple. We shall discuss only frames with joints lying in a plane, and we shall not consider displacements out of that plane.

In Fig. 50, we suppose the joint  $A$  fixed and the joint  $B$  constrained to slide on a horizontal line. Inspection shows that the whole frame is fixed by these conditions. In fact, the frame is a rigid body and is fixed when one of its points is fixed and another of its points constrained to move on a line. If one bar, for example  $CD$ , were removed, the frame would cease to be a rigid body. Hence it is called *just-rigid*. The following is the general definition: *A frame is just-rigid when the removal of any one of its bars destroys its rigidity.*

If an additional bar is inserted in a just-rigid frame, it becomes *over-rigid*. We shall deal only with just-rigid frames.

We shall now show that a just-rigid frame with  $j$  joints has  $2j - 3$  bars. Taking any axes in the plane of the frame, we denote the coordinates of the joints by  $(x_1,$



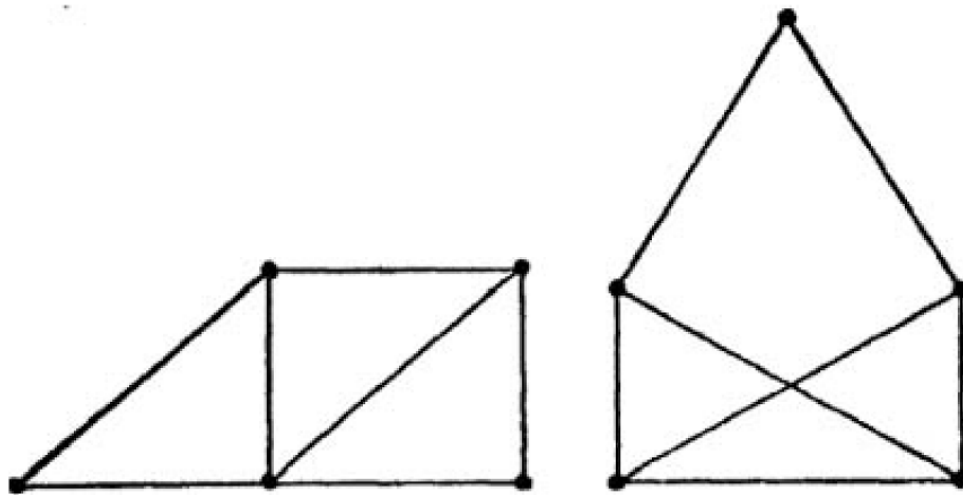


FIG. 53.—Frames with five joints and seven bars.

A simple way to build up a just-rigid frame is to start with a triangle and add two bars at a time. Since, in each operation, we add one joint and two bars, after  $p$  operations we have  $3 + p$  joints and  $3 + 2p$  bars; the identity

$$3 + 2p = 2(3 + p) - 3$$

shows that the condition for a just-rigid frame is satisfied. However, all just-rigid frames cannot be constructed in this way.

### Stresses in bars.

Suppose that a just-rigid frame is fixed by external constraints so that it cannot move. (The normal plan is to fix one joint and constrain another joint to move on a line, as in Fig. 50.) Now let external forces, or loads, be applied to some or all of the joints. Each bar is in equilibrium under two forces, the reactions at its ends. These two forces must be equal in magnitude and act in opposite senses along the bar. If the forces act away from one another (so that the bar tends to be torn in two), the bar is said to be in *tension*; if the forces act toward one another (so that the bar tends to buckle), the bar is said to be in *thrust*. The word *stress*

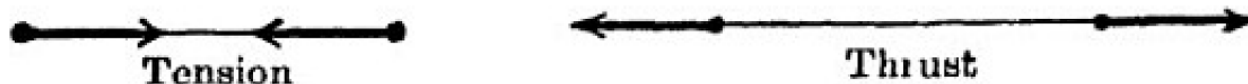


FIG. 54.—Reactions exerted by a bar on the joints at its ends.

is used to cover both cases. A plus sign is associated with tension and a minus sign with thrust. Thus, if we say that the stress in a bar is +3 tons, we mean that there is a tension of 3 tons in it; if the stress is -5 tons, we mean that there is a thrust of 5 tons. In Fig. 54 the arrows indicate forces exerted *on* the joints *by* the bars. The forces exerted on the bars by the joints act in the reverse directions.

For a frame in equilibrium, two problems arise:

- (i) to determine the external reactions at the supported joints;
- (ii) to determine the stresses in the bars.

The first problem is elementary. It is a question of the equilibrium of a system, as discussed in Secs. 2.3 and 2.4 and summarized in Sec. 2.6. It is with the second problem that we are concerned.

### Method of joints.

The following argument is general, but the reader may consider the frame shown in Fig. 50 as an example, the loads being indicated by arrows at the joints  $E$  and  $F$ . The loads are given, and the stresses are to be found. Each joint may be considered as a particle in equilibrium, under the action of a load (if any) and the reactions of the bars meeting there. (Since the joint is the system considered, this method is called the *method of joints*.) As the forces lie in a plane, there are two equations of equilibrium for each joint, and thus  $2j$  equations in all if the number of joints is  $j$ . These equations involve 3 unknown components of external reactions at the supports and a number of unknown stresses equal to the number of bars, i.e.,  $2j - 3$ . Thus the total number of unknowns is  $2j$ , and we have  $2j$  linear equations to find them. *Thus, in a just-rigid frame the problem of finding the external reactions at the supports and the stresses in the bars is a determinate problem, involving the solution of a number of simultaneous linear equations equal to twice the number of joints.*

If the frame were over-rigid, the number of unknowns would exceed the number of equations, and the problem would be indeterminate. We should have to consider the elastic properties of the bars.

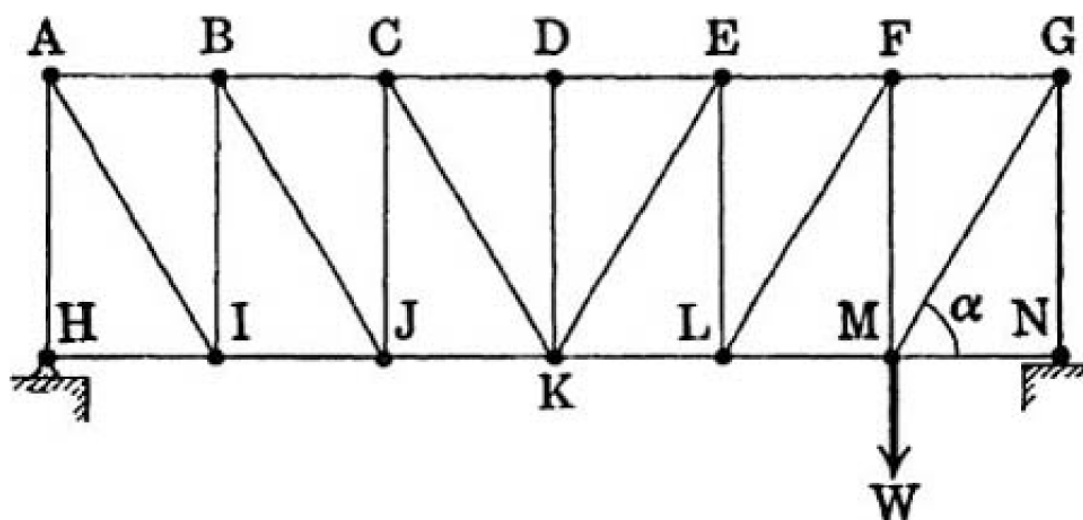


FIG. 55a.—A frame with 14 joints, supporting a load at  $M$ .

The problem of the just-rigid frame having been thus reduced to the solution of simultaneous linear equations, it might be thought that nothing remained to be said. However, the system of equations obtained in the manner described above



FIG. 55*b*.—Method of sections: the “system” is enclosed by the broken line.

equilibrium are applied. Up till now we have been thinking of a single bar, the whole frame, or a single joint as the system. But here we take a different approach, following the *method of sections*.

Figure 55*b* shows the same frame as that of Fig. 55*a*, with the same load. We wish to find the stresses in  $KL$ ,  $KE$ ,  $DE$ . We consider as a system the part of the frame enclosed within a curved line cutting the bars  $KL$ ,  $KE$ ,  $DE$ , but no others. This system is acted on by the following external forces:

- the load  $W$  at  $M$ ;
- the reaction  $R_N$  at  $N$ ;
- the stresses in  $KL$ ,  $KE$ ,  $DE$ .

Taking moments about  $E$ , we have

$$S_{KL} \cdot EL + W \cdot EF = R_N \cdot EG.$$

But  $R_N$  may be found by consideration of the equilibrium of the whole frame; hence

$$S_{KL} = \frac{2}{3}W \cot \alpha.$$

From consideration of the total vertical component of force, we have

$$S_{KE} = -\frac{1}{3}W \operatorname{cosec} \alpha;$$

and, from the total horizontal component,

$$S_{DE} = -S_{KL} - S_{KE} \cos \alpha = -\frac{1}{2}W \cot \alpha.$$

We note that the method would not have worked had the three bars cut by the section met in a point.

### Method of virtual work.

The method of virtual work may be applied to the problem just treated. To find  $S_{KL}$ , we suppose the bar  $KL$  removed and forces applied to the joints  $K$  and  $L$  equal to the (unknown) stress in  $KL$ . The frame is no longer rigid, but it is in equilibrium. Hence the virtual work done in an infinitesimal displacement is zero. For infinitesimal displacement, let us take a rotation about  $E$  of the right-hand portion of the frame. The only forces to do work are the load  $W$ , the reaction  $R_N$ , and the force at  $L$  replacing the stress  $S_{KL}$ . Equating the work done by them to zero, we obtain the expression for  $S_{KL}$  given above.

To get the stress in  $EK$ , we replace the bar  $KL$  and remove  $EK$ , at the same time applying to the joints  $E$  and  $K$  forces equal to the (unknown) stress in  $EK$ . Now we give a virtual displacement, holding the left-hand portion fixed. The right-hand side rises slightly with parallel displacement of its bars, the bar  $DE$  hinging at  $D$  and  $KL$  hinging at  $K$ . The only working forces are the load  $W$ , the reaction  $R_N$ , and the force at  $E$  replacing the stress  $S_{KE}$ . Equating to zero the work done, we find for  $S_{KE}$  the value given above.  $S_{DE}$  is found similarly without difficulty.

## Complex frames.

Frames constructed by adding successive pairs of bars to a basic triangular frame are called *simple* frames. Those so far discussed have been of this type. But there are also just-rigid frames which cannot be built up in this way; such frames are called *complex*. An example is shown in Fig. 56, in which the bars are supposed to cross without touching. The stresses may be found by solving the 12 equations of equilibrium of the joints, but that method is complicated. We cannot use the step-by-step method of joints, because there is no joint at which only two bars meet. We modify the method by assigning an unknown value to one of the stresses; we find the other stresses in terms of this one unknown by the conditions of equilibrium of the joints and finally, on closing the calculation, determine the unknown stress and hence all the stresses.

Let us work this out in the case shown in Fig. 56, in which the bars  $FE$ ,  $ED$ ,  $AD$ ,  $FC$  are inclined to the horizontal at  $45^\circ$ , and  $AB$ ,  $BC$  inclined to the horizontal at  $30^\circ$ . Write  $S_{EB} = S$ . Then

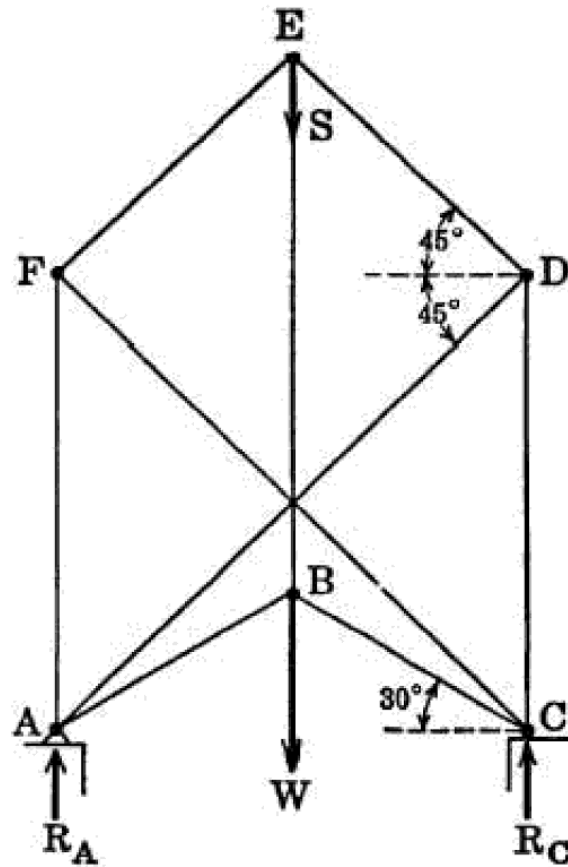


FIG. 56.—A complex frame.

$$\begin{aligned}
 &\text{at } E, & S_{ED} = S_{FE} = -S/\sqrt{2}, \\
 &\text{at } D, & S_{AD} = -S_{ED} = S/\sqrt{2}, \quad (= S_{FC}, \text{ by symmetry}), \\
 &\text{at } D, & S_{CD} = (S_{ED} - S_{AD})/\sqrt{2} = -S, \\
 &\text{at } C, & S_{CD} + S_{FC}/\sqrt{2} + S_{BC}/2 + R_C = 0, \\
 &\text{at } C, & S_{FC}/\sqrt{2} + S_{BC}\sqrt{3}/2 = 0.
 \end{aligned}$$

Elimination of  $S_{BC}$  from the last two equations gives

$$S_{CD}\sqrt{3} + S_{FC}(\sqrt{3} - 1)/\sqrt{2} + R_C\sqrt{3} = 0.$$

Since  $R_C = W/2$ ,  $S_{CD} = -S$ ,  $S_{FC} = S/\sqrt{2}$ , we obtain

$$S = \frac{1}{2}W(3 - \sqrt{3});$$

all the stresses are now easily written down.

### Concluding remarks.

The methods described above are adequate in simple cases, and the reduction of the problem of determining the stresses to the solution of a set of  $2j$  simultaneous linear equations is complete and satisfactory mathematically, although often complicated. When we have written down the equations, we know that we have given complete mathematical expression to all the conditions of

equilibrium and that the stresses can be found from the equations *provided they are consistent*.

It may happen that the equations of equilibrium are inconsistent; this occurs in the case of *critical forms*, of which an example is shown in Fig. 57. This frame is just-rigid ; but since the bars  $AB$ ,  $BC$  lie in a straight line, no stresses in them can give equilibrium of the joint  $B$ , when a load  $W$  is applied there. Such a frame would be an unsound engineering structure. Actually the joint  $B$  would be slightly depressed (owing to stretching of the bars), and there would be very great tensions in the bars  $AB$ ,  $BC$ .

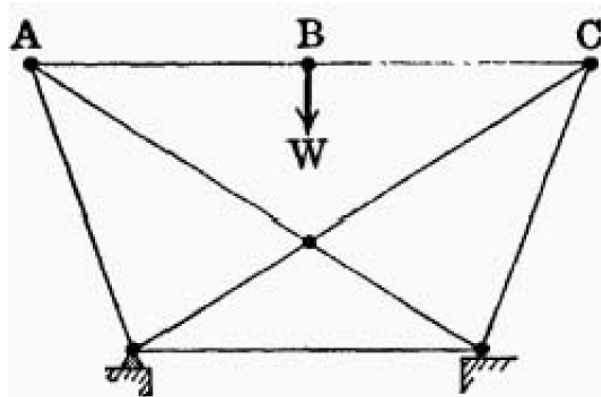


FIG. 57.—A critical form.

On account of its importance in engineering, the theory of frames has been elaborately developed. For a more complete account, with special reference to engineering problems, the reader is referred to S. Timoshenko and D. H. Young, *Engineering Mechanics* (McGraw-Hill Book Company, Inc., New York, 1940).

Most statical problems admit two methods of attack. On the one hand, we may reduce the problem to the solution of equations; this is the *analytical method*. On the other hand, we may represent forces by segments, and compound and resolve them by actual drawing; this is the *graphical method*. Each method has its advantages, but throughout this book we have preferred to use the analytical method, because it is easier to explain and has a wider range of application. For the graphical method in statics and its application to frames, the reader may consult for example H. Lamb, *Statics* (Cambridge University Press, 1928).

### 3.6. SUMMARY OF APPLICATIONS IN PLANE STATICS

#### I. Mass centers and centers of gravity.

(a) Formulas for the mass center:

$$(3.603) \quad k \frac{d^4 y}{dx^4} = -w.$$

$$(3.604) \quad M = k \frac{d^2 y}{dx^2}, \quad S = -\frac{dM}{dx}.$$

(d) Continuity conditions:  $y$ ,  $dy/dx$ ,  $d^2y/dx^2$  are continuous.

#### IV. Flexible cables.

(a) General formulas:

$$(3.605) \quad T \frac{dx}{ds} = H \quad (\text{a constant}); \quad \frac{d}{ds} \left( \frac{dy}{dx} \right) = \frac{w}{H}.$$

(b) Cable of suspension bridge hangs in a parabola.

(c) Common catenary:

$$(3.606) \quad y = c \left( \cosh \frac{x}{c} - 1 \right), \quad c = \frac{H}{w},$$

$$(3.607) \quad s = c \sinh \frac{x}{c},$$

$$(3.608) \quad s^2 + c^2 = (y + c)^2,$$

$$(3.609) \quad T = H + wy.$$

(d) Light cable in contact with a smooth curve:

$$(3.610) \quad T = \text{constant}, \quad N = \frac{T}{\rho}.$$

(e) Light cable in contact with a rough curve:

$$(3.611) \quad T = T_0 e^{\mu \theta} \quad (\text{for cable on point of slipping}).$$

#### V. Frames.

(a) Just-rigid frame:

$$(3.612) \quad b = 2j - 3$$

(b) Method of joints. Begin with a joint where only two bars meet.

(c) Method of sections. Section must not cut more than three bars, and these

three bars must not meet at a point.

(d) Method of virtual work. Remove a bar.

(e) Complex frames: (b) and (c) not applicable directly. Assume one stress  $S$ , and use (b).

### EXERCISES III

1. Find the mass center of a cubical box with no lid, the sides and bottom being made of the same thin material.

2. A ladder leans against a smooth wall, the lower end resting on a rough floor for which the coefficient of friction is  $\frac{1}{4}$ . Find the inclination of the ladder to the vertical, if it is just on the point of slipping.

3. A square frame is braced by two diagonal bars. One of these contains a turnbuckle, which is tightened until there is a tension  $T$  in the bar. Find the stresses in the other bars.

4. A man of weight  $W$  walks slowly along a light plank of length  $a$ , supported at its ends. Find the bending moment in the plank directly beneath his feet as a function of his distance from one end of the plank. Find also the shearing forces just in front of him and just behind him. Draw diagrams to show the senses of the bending moment and shearing forces.

5. Find the mass center of a wire bent into the form of an isosceles right-angled triangle.

6. A rod 4 ft. long rests on a rough floor against the smooth edge of a table of height 3 ft. If the rod is on the point of slipping when inclined at an angle of  $60^\circ$  to the horizontal, find the coefficient of friction.

7. A body of weight  $w$  rests on a rough inclined plane of inclination  $i$ , the coefficient of friction ( $\mu$ ) being greater than  $\tan i$ . Find the work done in slowly dragging the body a distance  $a$  up the plane and then dragging it back to the starting point, the applied force being in each case parallel to the plane.

8. A heavy cable rests in contact with a smooth curve in a vertical plane. Show that the difference in the tension at two points of the cable is proportional to the difference in level at these points.

9. Two light rings can slide on a rough horizontal rod. The rings are connected by a light inextensible string of length  $a$ , to the mid-point of which is attached a weight  $W$ . Show that the greatest distance between the rings, consistent with the equilibrium of the system, is

$$\mu a / \sqrt{1 + \mu^2},$$

where  $\mu$  is the coefficient of friction between either ring and the rod.



**10.** A heavy beam  $ABCD$ , of weight 2 lb. per ft., is supported horizontally by knife-edges at  $B$  and  $D$ . The beam is subjected to an additional vertical load of 20 lb. at  $C$ . If  $AB = BC = CD = 4$  ft., determine the shearing force and bending moment for all points of the beam.

**11.** A portion of a circular disk of radius  $r$  is cut off by a straight cut of length  $2c$ . Find the position of the mass center of the larger portion.

If  $r = 1$  ft.,  $c = 6$  in., calculate the distance of the mass center from the center of the circle.

**12.** A light cable connects two weights  $W, w$  ( $W > w$ ) and passes over a rough circular cylinder whose axis is horizontal.  $W$  rests on the ground, and  $w$  is suspended in the air. Find the least value of the coefficient of friction between the cylinder and the cable in order that  $W$  may be raised from the ground by slowly rotating the cylinder, and find expressions for the work done in turning the cylinder through one revolution (i) if  $W$  is raised, (ii) if  $W$  is not raised.

Find also the work done in turning the cylinder through one revolution in the opposite sense.

**13.** For the frame shown in Fig. 55*a*, take  $\alpha = 45^\circ$ , and find the stresses in all the bars. Make a sketch of the frame, marking with a double line each bar in which there is a thrust.

**14.** A uniform semicircular wire hangs on a rough peg, the line joining its extremities making an angle of  $45^\circ$  with the horizontal. If it is just on the point of slipping, find the coefficient of friction between the wire and the peg.

**15.** An elastic beam rests on three props, two being situated at the ends of the beam and at the same height, and the third at the middle point of the beam. Find the height of this central prop if the pressures on all three props are equal.

**16.** A cable 200 ft. long hangs between two points at the same height. The sag is 20 ft., and the tension at either point of suspension is 120 lb. wt. Find the total weight of the cable.

**17.** A uniform cable hangs across two smooth pegs at the same height, the ends hanging down vertically. If the free ends are each 12 ft. long and the tangent to the catenary at each peg makes an angle of  $60^\circ$  with the horizontal, find the total length of the cable.

**18.** Consider a frame as in Fig. 50, the triangles being equilateral. It is to carry a load  $2W$ , either as a single load at  $E$  or equally divided between  $E$  and  $F$ . In which case is there greater danger of collapse, it being assumed that collapse is due to a thrust in a bar exceeding some definite value, the same for all bars?

**19.** Four rods each of length  $a$  and weight  $w$  are smoothly jointed together to form a rhombus  $ABCD$ , which is kept in shape by a light rod  $BD$ . The angle  $BAD$  is  $60^\circ$ , and the rhombus is suspended in a vertical plane from  $A$ . Find the tension or thrust in  $BD$  and the magnitude and direction of the force exerted by the joint



$$(4.102) \quad \frac{d\mathbf{i}}{d\phi} = \mathbf{j}, \quad \frac{d\mathbf{j}}{d\phi} = -\mathbf{i}.$$

To find the acceleration  $\mathbf{f}$ , we differentiate (4.101); this gives

$$(4.103) \quad \mathbf{f} = \frac{d\mathbf{q}}{dt} = \mathbf{i} \frac{d^2s}{dt^2} + \frac{d\mathbf{i}}{d\phi} \frac{d\phi}{ds} \left( \frac{ds}{dt} \right)^2.$$

Hence, by (4.102) and the fact that the radius of curvature of  $C$  is  $\rho = ds/d\phi$ , we have

$$(4.104) \quad \mathbf{f} = \mathbf{i} \frac{dq}{dt} + \mathbf{j} \frac{q^2}{\rho};$$

or, in words, *the acceleration of a moving particle has a component  $dq/dt$  along the tangent and a component  $q^2/\rho$  along the normal to the path.* The tangential component may also be expressed in the form  $q dq/ds$ .

It is easily seen that the normal component of acceleration always points to the concave side of the path.

As an example, consider a particle traveling in a circle of radius  $r$  with a speed  $q$  which is (a) constant, (b) proportional to  $t$ . In case (a), the acceleration vector is directed inward along the radius and has a magnitude  $q^2/r$ ; in case (b), the acceleration vector has a constant component along the tangent and a component along the radius which varies as  $t^2$ .

## Radial and transverse components.

Consider a particle  $P$  moving in a plane, its position being described by polar coordinates  $r, \theta$  (Fig. 59). Let  $\mathbf{i}$  be the unit vector along  $OP$  and  $\mathbf{j}$  the unit vector perpendicular to  $\mathbf{i}$ , drawn in the sense shown. We note that

$$(4.105) \quad \frac{d\mathbf{i}}{d\theta} = \mathbf{j}, \quad \frac{d\mathbf{j}}{d\theta} = -\mathbf{i}.$$

We have then  $\mathbf{r} = r\mathbf{i}$ , and the velocity is

$$(4.106) \quad \mathbf{q} = \dot{\mathbf{r}} = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j},$$

the dot indicating  $d/dt$ . Thus  $(\dot{r}, r\dot{\theta})$  are the components of velocity along and perpendicular to the radius vector. For the acceleration we find, on differentiating (4.106) and using (4.105),

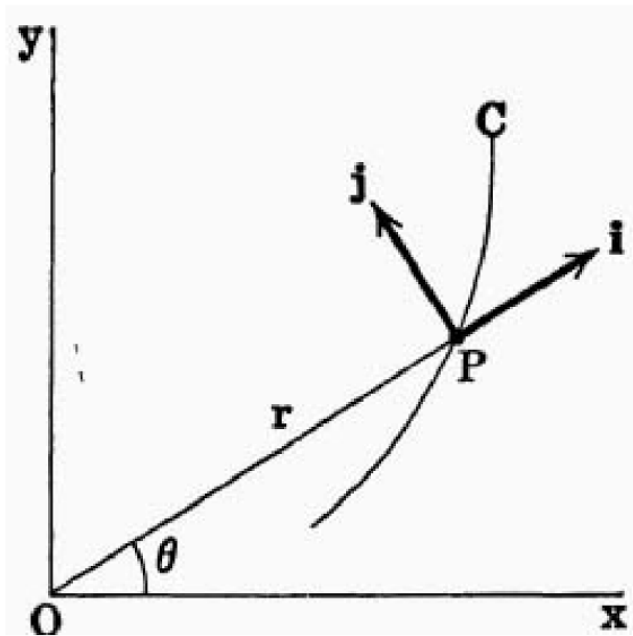


FIG. 59.—Resolution along and perpendicular to radius vector.

$$(4.107) \quad \mathbf{f} = \dot{\mathbf{q}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i} + \frac{1}{r} \frac{d}{dt} (r^2\dot{\theta})\mathbf{j}.$$

Thus

$$\ddot{r} - r\dot{\theta}^2, \quad \frac{1}{r} \frac{d}{dt} (r^2\dot{\theta})$$

are the components of acceleration along and perpendicular to the radius vector.

It is usual to call the components in the directions  $\mathbf{i}$  and  $\mathbf{j}$  the *radial* and *transverse* components, respectively.

## The hodograph.

It is easy to form an intuitive picture of the velocity of a particle; we have merely to visualize the small displacement it receives in a small time and then imagine that small displacement greatly magnified without change of direction. But it is much more difficult to form an intuitive picture of the acceleration. The *hodograph* is a device to facilitate this. Figure 60a shows the path  $C$  of a particle, with its velocity  $\mathbf{q}$  and acceleration  $\mathbf{f}$  at the position  $P$ . Imagine now a fictitious particle  $P'$  moving in the plane (Fig. 60b) with a motion correlated to the motion of  $P$  by the following rule: *the position vector of  $P'$ , relative to some chosen origin  $O'$ , is equal to the velocity of  $P$* . The path described by  $P'$  is called the *hodograph* of the motion of  $P$ .

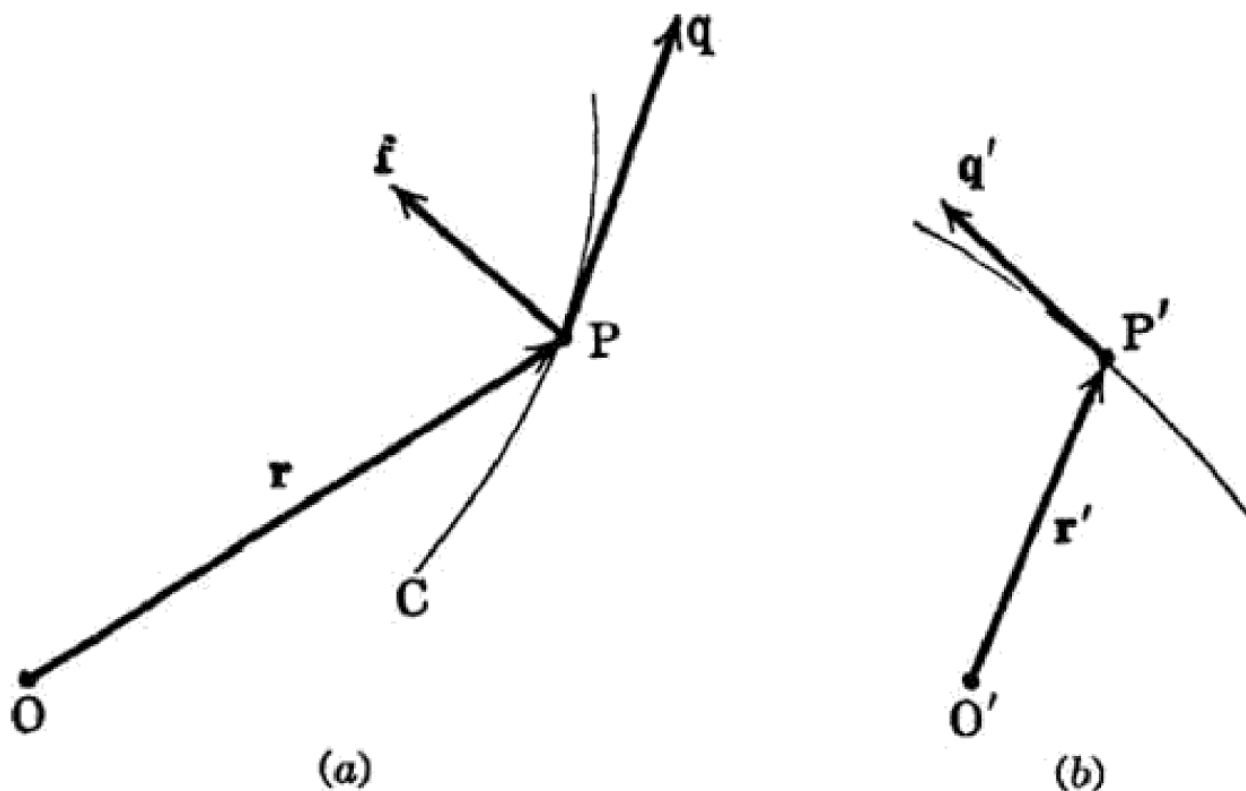


FIG. 60.—(a) Motion. (b) Hodograph.

Denoting by  $r'$ ,  $q'$  the position vector and velocity of  $P'$ , we have

$$(4.108) \quad r' = q.$$

Hence, on differentiation,

$$(4.109) \quad q' = f;$$

in words, *the velocity in the hodograph is equal to the acceleration in the actual motion.*

*Exercise.* Verify the following statements:

- (i) If a particle has an acceleration which is constant in magnitude and direction, the hodograph is a straight line described with constant speed.
- (ii) If a particle moves in a circle with constant speed, the hodograph is a circle described with constant speed.

## 4.2. MOTION OF A RIGID BODY PARALLEL TO A FIXED PLANE

### Description of the motion.

As already remarked in Sec. 2.4, the motion of a rigid body parallel to a fixed plane is completely described by the motion of the representative lamina, i.e., the section of the body by the plane. We may therefore confine our attention to the representative lamina. We also discussed in Sec. 2.4 the general

coordinates of  $A$  and  $(x, y)$  those of  $P$ , both measured on fixed axes  $Oxy$ . Then the infinitesimal displacement of  $P$  in the time interval  $dt$  has components

$$(4.201) \quad \begin{cases} dx = u_A dt - (y - b)\omega dt, \\ dy = v_A dt + (x - a)\omega dt, \end{cases}$$

where  $u_A, v_A$  are the components of  $q_A$ . Thus the velocity of  $P$  has components

$$(4.202) \quad \begin{cases} u = u_A - (y - b)\omega, \\ v = v_A + (x - a)\omega. \end{cases}$$

### Instantaneous center.

At any instant, there is just one point of a moving lamina which has no velocity. Its coordinates are found from (4.202), on putting  $u = v = 0$ ; they are

$$(4.203) \quad x = a - v_A/\omega, \quad y = b + u_A/\omega.$$

This point is called the *instantaneous center*. Under one exceptional condition no instantaneous center exists, namely, when  $\omega = 0$ . We may then say that the instantaneous center is at infinity.

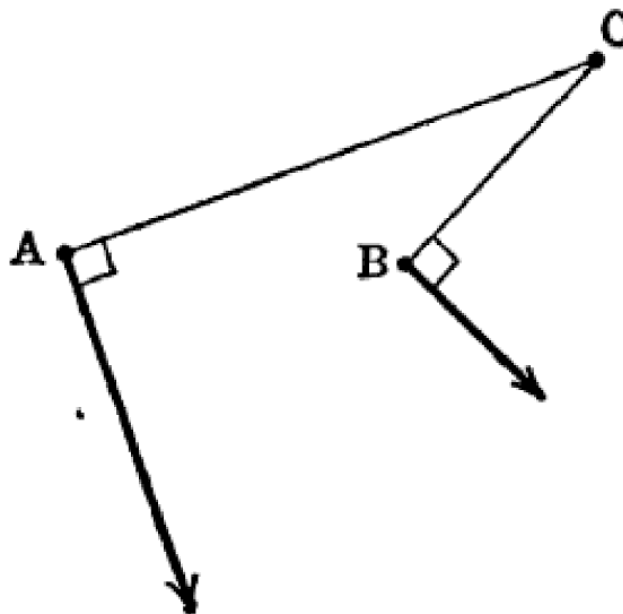


FIG. 62.—Determination of the instantaneous center from the velocities of two points.



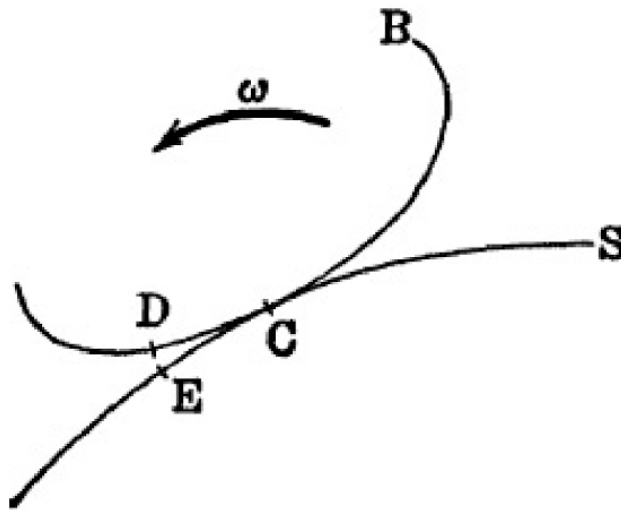


FIG. 63.—The body centrode  $B$  rolls on the space centrode  $S$ .

Once the instantaneous center  $C$  is known, it is very easy to visualize what happens to the lamina in a small interval of time  $dt$ : the lamina rotates about  $C$  through a small angle  $\omega dt$ . This fact enables us to find  $C$  when the directions of the velocities of two points  $A$  and  $B$  of the lamina are known. For, since the lamina is turning about  $C$  at the instant, the velocity of any point  $P$  is perpendicular to  $CP$ . Hence,  $C$  is located at the intersection of the lines drawn through  $A$  and  $B$  perpendicular to the velocities of those points (Fig. 62).

From the definition of the rolling of one curve on another, given in Sec. 2.4, it is now clear that *a moving curve rolls on a fixed curve when the curves touch and the instantaneous center of the moving curve is at the point of contact*. In fact, this statement may be taken as a definition of rolling, instead of that given in Sec. 2.4. If both curves are in motion, we define rolling by the conditions that the curves touch and that the instantaneous velocities of the two particles at the point of contact (one on each curve) are equal to one another.

As a lamina moves, the instantaneous center  $C$  moves in the fixed plane; the curve described by it is called the *space centrode* ( $S$ ). But  $C$  also moves in the lamina; the curve described by  $C$  in the lamina is called the *body centrode* ( $B$ ). At any instant,  $S$  and  $B$  have the point  $C$  in common, and  $B$  is turning about  $C$ , since  $B$  is carried along with the lamina. The situation at time  $t$  is shown in Fig. 63. A little later, at time  $t + dt$ , a point  $D$  of  $B$  has moved into coincidence with a point  $E$  of  $S$  to form the new instantaneous center, and this is done by turning  $B$  about  $C$  through the small angle  $\omega dt$ . Hence, it is evident that  $B$  cannot cut  $S$  at a finite angle; therefore  $B$  touches  $S$  at  $C$ , and since  $C$  is the instantaneous center of  $B$ , we have the following result: *The body centrode rolls on the space centrode*.

*Exercise.* Verify the following statements:

- (i) When a wheel rolls on a track, the space centrode is the track itself and the body

centrode the circumference of the wheel.

(ii) When a rod of length  $2a$  slides with its extremities on two lines which intersect at right angles, the body centrode is a circle of radius  $a$  and the space centrode a circle of radius  $2a$ .

*Example.* As an example of our analysis of the motion of a rigid body, let us consider two wheels  $W_1$  and  $W_2$  of radii  $a_1$  and  $a_2$  respectively, lying in a plane. Their centers are connected by a rod  $R$  of length  $a_1 + a_2$ , and the wheels engage without slipping. If we fix the center of  $W_1$ , then  $W_1$  and  $R$  can turn independently about this center, and  $W_2$  will roll on  $W_1$ . Each of the three bodies  $W_1$ ,  $W_2$ ,  $R$  has an angular velocity, say  $\omega_1$ ,  $\omega_2$ ,  $\Omega$ . These three angular velocities are not independent; let us find the relation connecting them.

The particles of  $W_1$  and  $W_2$  at their point of contact have the same velocity. We can find two different expressions for this common velocity; equating them, we obtain the required relation. Since  $W_1$  turns about its center with angular velocity  $\omega_1$ , the velocity of its particle at the point of contact is tangential and of magnitude  $a_1\omega_1$ . The wheel  $W_2$  has a motion which may be described by means of a base point taken at its center. The velocity of this base point is perpendicular to  $R$  and of magnitude  $(a_1 + a_2)\Omega$ . Hence the velocity of the particle of  $W_2$  at the point of contact with  $W_1$  is tangential and of magnitude

$$(\alpha_1 + \alpha_2)\Omega - \alpha_2\omega_2.$$

Therefore we have, as the required relation in symmetric form,

$$\alpha_1\omega_1 + \alpha_2\omega_2 = (\alpha_1 + \alpha_2)\Omega.$$

As an alternative method of finding  $\omega_2$  when  $\omega_1$  and  $\Omega$  are given, the following general method of finding angular velocity may be used. Take two particles of the body, say  $A$  and  $B$ . Resolve their velocities perpendicular to  $AB$ . The difference of these component velocities, divided by  $AB$ , is the required angular velocity. The proof of this is left as an exercise.

### 4.3. SUMMARY OF PLANE KINEMATICS

#### I. Kinematics of a particle.

(a) Components of velocity and acceleration:

perpendicular to one another.

5. Starting from

$$x = r \cos \theta, y = r \sin \theta,$$

calculate  $\ddot{x}$  and  $\ddot{y}$ . Hence, by resolving the acceleration vector along and perpendicular to the radius vector, establish the formula (4.107).

6. A wheel of radius  $a$  rolls without slipping along a straight road. If the center of the wheel has a uniform velocity  $v$ , find at any instant the velocity and acceleration of the two points of the rim which are at a height  $h$  above the road. Examine in particular the cases  $h = 0$ ,  $h = 2a$ .

7. Is it possible for a particle to move in a circle and have a hodograph which is a straight line? Give reasons for your answer.

8. A wheel of radius  $a$  rolls along a straight track, the center having a constant acceleration  $f$ . Show that there is, at any instant, just one point of the wheel with no acceleration; find its position relative to the center of the wheel.

9. A rectangular plate  $ABCD$  moves in its plane with constant angular velocity  $\omega$ . At a given instant the point  $A$  has a velocity of magnitude  $V$  along the diagonal  $AC$ . Find the velocity of  $B$  at this instant in terms of  $V$ ,  $\omega$ , and the dimensions of the rectangle.

10. A uniform circular hoop of radius  $a$  rolls on the outer rim of a fixed wheel of radius  $b$ , the hoop and the wheel being coplanar. If the angular velocity  $\omega$  of the hoop is constant, find

(i) the velocity and acceleration of the center of the hoop;

(ii) the acceleration of that point of the hoop which is at the greatest distance from the center of the wheel.

11. A motorboat experiences a resistance proportional to the square of the speed. The engine is switched off when the speed is 50 ft. per sec. When the boat has moved through a distance of 60 ft., its speed has been reduced to 20 ft. per sec. Find (to the nearest foot) the total distance traversed when the speed has been reduced to 10 ft. per sec.

12. A point  $A$  has a uniform circular motion about a fixed point  $O$  with angular velocity  $m$ . A point  $B$  has a uniform circular motion about  $A$  with angular velocity  $n$ . What relation connects  $m$  and  $n$  if the acceleration of  $B$  is always directed toward  $O$ ?

13. A circular ring of radius  $b$  turns in its plane about its center with constant angular velocity  $\Omega$ . A second circular ring of radius  $a$  ( $< b$ ) rolls in the same plane on the inner side of the first ring. The angular velocity of the center

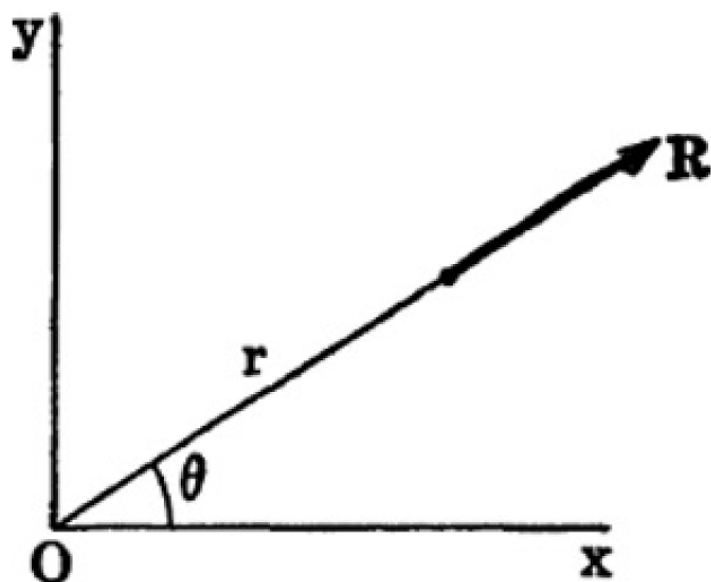


FIG. 65.—A central force.

$$(5.104) \quad m(\ddot{r} - r\dot{\theta}^2) = R, \quad \frac{d}{dt}(r^2\dot{\theta}) = 0.$$

Equations (5.102), (5.103), and (5.104) are all very useful forms of the equations of motion of a particle.

*Exercise.* Referring to (3.122), write down the equations of motion of a particle in the earth's gravitational field, using (i) polar coordinates and (ii) rectangular Cartesians.

### Principle of angular momentum.

The *momentum* of a particle of mass  $m$ , moving with velocity  $\mathbf{q}$ , is defined as the vector  $m\mathbf{q}$ . (This is sometimes called *linear momentum*, to distinguish it from angular momentum, defined below.) The components of momentum for motion in a plane are

$$m\dot{x}, \quad m\dot{y},$$

where  $Oxy$  are rectangular axes in the plane. Since momentum is a vector, it has a moment about any point  $A$  in the plane; this moment is called *moment of momentum* or *angular momentum* about  $A$ . In [Chap. II](#), we discussed moments of vectors; we saw that the moment of a vector is the sum of the moments of its components, and so by (2.303) the angular momentum of a moving particle about the origin is

$$(5.105) \quad h = m(xy\dot{y} - y\dot{x}).$$

If, instead of resolving the momentum vector along the axes, we resolve it along and perpendicular to the radius vector drawn from the origin, we obtain components [cf. (4.106)]

$$m\dot{r}, \quad mr\dot{\theta}.$$

The former component has no moment about the origin; hence

$$(5.106) \quad h = mr^2\dot{\theta}.$$

Consider now a particle moving in a plane under the action of a force with components  $X, Y$ . The rate of change of angular momentum about the origin is

$$\dot{h} = m(xy\dot{y} - yx\dot{x}) = xY - yX = N,$$

where  $N$  is the moment of the force about the origin. Hence we have the *principle of angular momentum*: For a particle moving in a plane the rate of change of angular momentum about any fixed point in the plane is equal to the moment of the force about that point.

We note that, in the case of a central force, the second equation of (5.104) is equivalent to the statement that the angular momentum about the origin is constant.

Linear momentum, being the product of mass and velocity, has the dimensions  $[MLT^{-1}]$ . In the c g s. system, it is measured in gm. cm. sec.<sup>-1</sup>; in the f p.s. system in lb. ft. sec.<sup>-1</sup>. On account of the equivalence of dimensions (see Appendix), these units may also be called dyne sec. and poundal sec., respectively. Angular momentum has the dimensions  $[ML^2T^{-1}]$  and is measured in gm. cm.<sup>2</sup> sec.<sup>-1</sup> or lb. ft.<sup>2</sup> sec.<sup>-1</sup>.

### Principle of energy.

The *kinetic energy*<sup>\*</sup> of a particle of mass  $m$  moving with velocity  $q$  is defined to be  $\frac{1}{2}mq^2$ . It will be denoted by  $T$ . Thus, for a particle moving in space,

$$(5.107) \quad T = \frac{1}{2}mq^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

The rate of change of kinetic energy is

$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = X\dot{x} + Y\dot{y} + Z\dot{z},$$

where  $X, Y, Z$  are the components of the force acting on the particle. The increase in kinetic energy in the interval  $(t_0, t_1)$

is therefore

$$T_1 - T_0 = \int_{t_0}^{t_1} (X\dot{x} + Y\dot{y} + Z\dot{z}) dt.$$

Let  $W$  denote the work done by the force during this time interval. By (2.403)

the work done in an infinitesimal displacement is

$$dW = X dx + Y dy + Z dz = (X\dot{x} + Y\dot{y} + Z\dot{z}) dt,$$

and so

$$W = \int_{t_0}^{t_1} (X\dot{x} + Y\dot{y} + Z\dot{z}) dt.$$

Hence

$$(5.108) \quad T_1 - T_0 = W.$$

This establishes the *principle of energy*: *The increase in kinetic energy is equal to the work done by the force.*

Differentiating (5.108) with respect to  $t_1$  and then dropping the subscript 1, we have

$$(5.109) \quad \dot{T} = \dot{W};$$

in words, *the rate of increase of kinetic energy equals the rate of working of the force.*

If the particle moves in a conservative field of force with potential energy  $V$ , then, as in (2.419),

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}.$$

Then

$$W = \int_{t_0}^{t_1} (X\dot{x} + Y\dot{y} + Z\dot{z}) dt = - \int_{t_0}^{t_1} \dot{V} dt = -V_1 + V_0,$$

where  $V_1, V_0$  are the potential energies at times  $t_1, t_0$ , respectively. Comparison with (5.108) gives

$$T_1 - T_0 = -V_1 + V_0, \quad T_1 + V_1 = T_0 + V_0.$$

Hence, in general,

$$(5.110) \quad T + V = E,$$

where  $E$  is a constant, called the *total energy*. Thus *the sum of the kinetic and potential energies is constant*. This is called *the principle of the conservation of*



$$(5.204) \quad \sum_{i=1}^n \mathbf{P}_i' = 0,$$

because this summation consists of vectors which are formed from pairs of forces that are equal and opposite. Hence, (5.203) gives

$$(5.205) \quad \dot{\mathbf{M}} = \sum_{i=1}^n \mathbf{P}_i,$$

which proves the principle of linear momentum.

The last equation may be written

$$(5.206) \quad \dot{\mathbf{M}} = \mathbf{F},$$

where  $\mathbf{F}$  is the vector sum of the external forces.

We shall now prove the *law of motion of the mass center*: *The mass center of a system moves like a particle, having a mass equal to the total mass of the system, acted on by a force equal to the vector sum of the external forces acting on the system.*

From (3.101) it follows that the velocity of the mass center of a system of particles is

$$(5.207) \quad \bar{\mathbf{q}} = \sum_{i=1}^n m_i \mathbf{q}_i / m,$$

where  $m$  is the total mass of the system. Thus if  $\bar{\mathbf{f}}$  is the acceleration of the mass center, we have

$$(5.208) \quad m\bar{\mathbf{f}} = m\dot{\bar{\mathbf{q}}} = \sum_{i=1}^n m_i \mathbf{f}_i = \dot{\mathbf{M}},$$

and so, by (5.206),

$$(5.209) \quad m\bar{\mathbf{f}} = \mathbf{F}.$$

This is the equation of motion of a particle of mass  $m$  acted on by a force  $\mathbf{F}$ , and so the law is established.

We note that, by (5.207),

$$(5.210) \quad m\bar{\mathbf{q}} = \sum_{i=1}^n m_i \mathbf{q}_i,$$

so that the linear momentum of the fictitious particle moving with the mass center is equal to the linear momentum of the system.

The conclusions to be drawn from the preceding principles are simple and interesting when the vector sum of the external forces is zero. Then the linear momentum of the system remains constant, and its mass center travels in a straight line with constant speed. This is true in particular for a stick sliding on a frozen pond. As for the solar system, we see that any Newtonian frame of reference must be such that the mass center of the solar system has a constant velocity relative to it.

### **Principle of angular momentum; motion relative to mass center.**

The *angular momentum* of a system about a line (or about a point in its plane if the system is confined to a plane) is defined as the sum of the angular momenta of the particles composing it. Thus if the particle of mass  $m_i$  has coordinates  $x_i, y_i, z_i$  and velocity components  $\dot{x}_i, \dot{y}_i, \dot{z}_i$ , the angular momentum of the system about  $Oz$  is

$$(5.211) \quad h = \sum_{i=1}^n m_i (x_i \dot{y}_i - y_i \dot{x}_i).$$

The rate of change of angular momentum about  $Oz$  is then

$$(5.212) \quad \dot{h} = \sum_{i=1}^n m_i (x_i \ddot{y}_i - y_i \ddot{x}_i).$$

But

$$m_i \ddot{x}_i = X_i + X'_i, \quad m_i \ddot{y}_i = Y_i + Y'_i,$$

where  $X_i, Y_i$  are the components of external force acting on the particle and  $X'_i, Y'_i$  the components of internal force. Thus,

$$(5.213) \quad \begin{aligned} \dot{h} &= \sum_{i=1}^n (x_i Y_i - y_i X_i) + \sum_{i=1}^n (x_i Y'_i - y_i X'_i) \\ &= N + N', \end{aligned}$$

where  $N$  is the total moment about  $Oz$  of all the external forces acting on the system and  $N'$  the total moment of all the internal forces. But since the internal forces occur in balanced pairs, their total moment is zero; hence

$$(5.214) \quad \dot{h} = N.$$

This equation expresses the *principle of angular momentum*: *The rate of change of the angular momentum of a system about a fixed line is equal to the total moment of the external forces about that line.*

If we think, for example, of a man on a trapeze, the only-external forces are (i) gravity and (ii) a reaction at the point of suspension. But the latter has no moment about the point of suspension. Hence the rate of change of angular momentum about the point of suspension is equal to the moment of the gravitational forces about that point.

Let  $\bar{x}, \bar{y}, \bar{z}$  be the coordinates of the mass center of a system and let  $x'_i, y'_i, z'_i$  be the coordinates of the  $i$ th particle relative to the mass center, so that

$$(5.215) \quad x_i = \bar{x} + x'_i, \quad y_i = \bar{y} + y'_i.$$

The components of velocity of the particle relative to the mass center are  $\dot{x}'_i, \dot{y}'_i, \dot{z}'_i$ , and so the angular momentum relative to a line through the mass center parallel to  $Oz$  is

$$(5.216) \quad h = \sum_{i=1}^n m_i (x'_i \dot{y}'_i - y'_i \dot{x}'_i),$$

where we use, in computing angular momentum, the velocities relative to the mass center. We shall refer to this briefly as the *angular momentum relative to the mass center*.

From (5.216), we obtain

$$(5.217) \quad \dot{h} = \sum_{i=1}^n m_i (x'_i \ddot{y}'_i - y'_i \ddot{x}'_i),$$

and hence by (5.215), differentiated twice,

$$(5.218) \quad \begin{aligned} \dot{h} &= \sum_{i=1}^n m_i [x'_i (\ddot{y}_i - \ddot{\bar{y}}) - y'_i (\ddot{x}_i - \ddot{\bar{x}})] \\ &= -\left(\sum_{i=1}^n m_i x'_i\right) \ddot{\bar{y}} + \left(\sum_{i=1}^n m_i y'_i\right) \ddot{\bar{x}} \\ &\quad + \sum_{i=1}^n (x'_i Y_i - y'_i X_i) + \sum_{i=1}^n (x'_i Y'_i - y'_i X'_i); \end{aligned}$$

as before,  $X_i, Y_i$  are components of external force and  $X'_i, Y'_i$  components of internal force. But, from the defining property of the mass center, we have

Then,

$$\begin{aligned}
 (5.221) \quad \dot{T} &= \sum_{i=1}^n m_i (\dot{x}_i \ddot{x}_i + \dot{y}_i \ddot{y}_i + \dot{z}_i \ddot{z}_i) \\
 &= \sum_{i=1}^n (X_i \dot{x}_i + Y_i \dot{y}_i + Z_i \dot{z}_i),
 \end{aligned}$$

where  $X_i, Y_i, Z_i$  are the components of the total force, external and internal, acting on the  $i$ th particle. Thus, if  $W$  is the work done by the forces from time  $t_0$  to time  $t$ , we have

$$(5.222) \quad \dot{T} = \dot{W}.$$

This is formally the same as (5.109), but here we are considering a system instead of a single particle. For a system, the *principle of energy* takes the following form: *The rate of change of kinetic energy of a system is equal to the rate of working of all the forces, external and internal.*

There is a sharp difference between the principles of linear and angular momentum on the one hand and the principle of energy on the other. In the principles of momentum the internal forces are eliminated; in the principle of energy they are not eliminated, except in the special case where they do no work and so contribute nothing to  $\dot{W}$ . In our idealized mathematical models, consisting of rigid bodies with smooth contacts, no work is done by the internal forces, and so they disappear from the principle of energy. In cases of collision, however, work may be done by the internal forces (see [Chap. VIII](#)); that is because, in such cases, it is impossible to regard the bodies as absolutely rigid.

When the system is conservative, with potential energy  $V$ , we have  $\dot{W} = -\dot{V}$  by (2.416); then (5.222) leads to the *principle of the conservation of energy*

$$(5.223) \quad T + V = E,$$

where  $E$  is the constant total energy.

### **D'Alembert's principle.**

The principle about to be discussed adds nothing essential to the principles already given, but it is interesting as an alternative expression.

We have regarded “force” as a primitive concept in mechanics, and we shall not abandon that point of view. One must guard against logical confusion in accepting the following definitions, in which we use the conventional terms.

For a slightly less accurate agreement in the case of experiments on the earth, we may take the earth itself as frame of reference.

We now raise the question: Knowing that a body behaves relative to a Newtonian frame of reference in accordance with the laws and principles discussed earlier, how does a body appear to behave when viewed from a frame of reference moving relative to the Newtonian frame?

### Frames of reference with uniform translational velocity.

Let  $S$  be a Newtonian frame of reference and  $S'$  a frame of reference which has, relative to  $S$ , a uniform (i.e., unaccelerated) translational motion. (If  $S$  is the earth's surface,  $S'$  might be a train running smoothly on straight tracks at constant speed.) We shall consider only two dimensions, but the argument can be extended to space immediately.

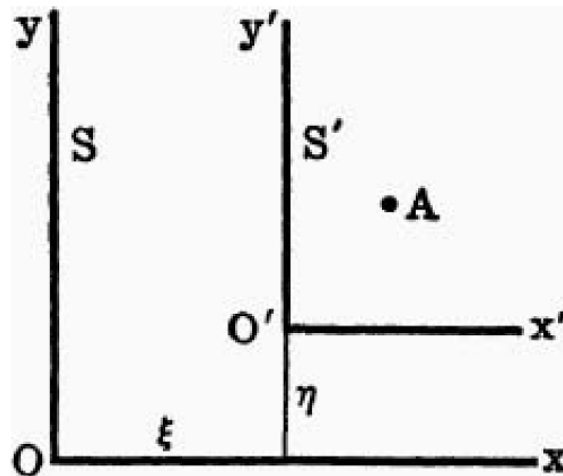


FIG. 66.—Frames of reference in relative motion without rotation.

In  $S$  we take axes  $Oxy$  and in  $S'$  we take parallel axes  $O'x'y'$  (Fig. 66). Let  $\xi$ ,  $\eta$  be the coordinates of  $O'$  relative to  $O$ . Then

$$(5.301) \quad \dot{\xi} = u_0, \quad \dot{\eta} = v_0,$$

where  $u_0$ ,  $v_0$  are the constant components of the velocity of  $S'$  relative to  $S$ . Let  $A$  be the position of any moving particle; it has coordinates  $(x, y)$  relative to  $Oxy$  and coordinates  $(x', y')$  relative to  $O'x'y'$ . These coordinates are connected by the relations

$$x = x' + \xi, \quad y = y' + \eta,$$

and, on differentiation,

$$(5.302) \quad \dot{x} = \dot{x}' + \dot{\xi}, \quad \dot{y} = \dot{y}' + \dot{\eta}.$$

Newtonian frame in which the mass center of the solar system is at rest. This is, in fact, the astronomical frame, to which we have referred before.

If the earth is regarded as a satisfactory Newtonian frame of reference, then we must regard as equally satisfactory the interior of any vehicle which moves over the earth with constant velocity. This is in accordance with common experience: we are not conscious of the smooth uniform motion of a train when we are traveling in it; we become conscious of the motion only when the train lurches or brakes or rounds a corner.

### Frames of reference with translational acceleration.

Let us now suppose that  $S$  is a Newtonian frame and that  $S'$  has relative to it a translational motion with constant acceleration. Then (cf. Fig. 66), we have

$$(5.307) \quad \xi = \alpha_0, \quad \ddot{\eta} = \beta_0,$$

where  $\alpha_0, \beta_0$  are constants. The relations (5.302) hold in this case also, and differentiation gives

$$(5.308) \quad \dot{x} = x' + \alpha_0, \quad \dot{y} = y' + \beta_0.$$

Let  $\mathbf{f}, \mathbf{f}'$  denote, respectively, the accelerations of  $A$  relative to  $S$  and  $S'$ , and let  $\mathbf{f}_0$  denote the acceleration of  $S'$  relative to  $S$ ; then (5.308) may be written

$$(5.309) \quad \mathbf{f} = \mathbf{f}' + \mathbf{f}_0.$$

This is called the *law of composition of accelerations*.

The equation of motion (5.305) now leads to

$$(5.310) \quad m\mathbf{f}' = \mathbf{P} - m\mathbf{f}_0.$$

Thus the Newtonian law of motion does *not* hold relative to  $S'$ . But we can say that the Newtonian law holds provided that we add to the *true* force  $\mathbf{P}$  a *fictitious* force— $m\mathbf{f}_0$ .

As an illustration, consider an elevator descending with constant acceleration  $f_0$ . Relative to the elevator, everything takes place as if the elevator were at rest and every particle experienced an upward force  $mf_0$ , where  $m$  is the mass of the particle, in addition to the downward force  $mg$ . These fictitious forces alter the reactions among the particles constituting the human body, and so we are conscious of an acceleration, even though we cannot look outside our frame of reference.

### Frames of reference rotating with constant angular velocity.

Let us now suppose that  $S$  is a Newtonian frame of reference and  $S'$  a frame of



greater than the force of gravity.

## Static effects of the earth's rotation.

In Sec. 3.1 we gave an introductory discussion of the force of gravity and the weight of a body near the earth's surface, leaving the earth's rotation out of account, i.e., treating the earth as a Newtonian frame. We now see that it may indeed be so treated provided that the proper fictitious forces are added. If we deal only with static problems, i.e., those in which the system is at rest relative to the earth, there is no Coriolis force, and so the only fictitious force is centrifugal. The "weight" of a particle is the resultant of the force of gravity and the

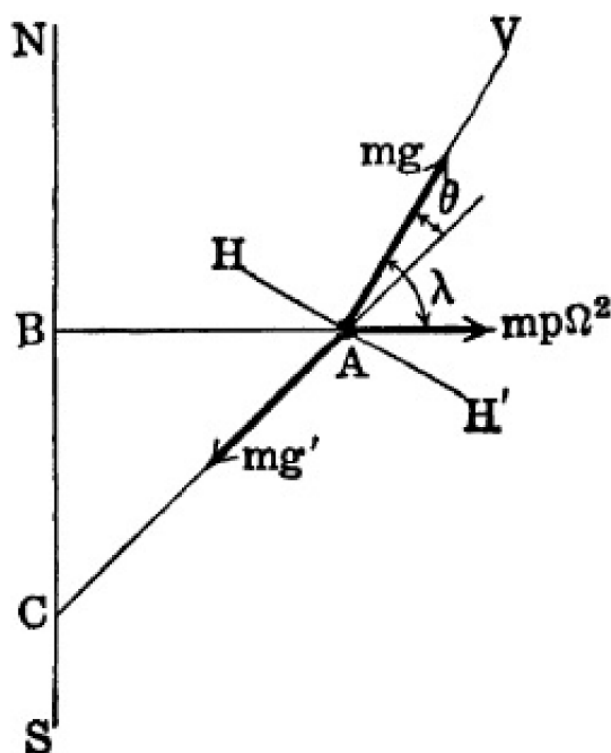


FIG. 70.—Plumb line on the rotating earth.

centrifugal force, instead of being merely the force of gravity alone. Thus the weight of a particle is proportional to its mass, and the theory of Sec. 3.1 is valid provided that we understand by  $mg$  the weight as just defined.

We shall now bring our theory still closer to reality by taking a more accurate model of the earth. As a first crude approximation, the earth may be regarded as a sphere of radius  $R$ , where  $R = 3960$  miles. More accurately, it is an oblate spheroid with an equatorial radius of 3963 miles and a polar radius of 3950 miles. This is the model which we shall accept for the present discussion, and we shall assume that the model rotates about its polar axis with constant angular velocity  $\Omega$ .

In Fig. 70,  $SN$  is the earth's axis,  $A$  any point on its surface, and  $AB$  the

perpendicular dropped on  $SN$ . The gravitational attraction of the earth on a particle at  $A$  acts along some line  $AC$  which intersects  $SN$ ; its magnitude is proportional to the mass  $m$  of the particle, and we shall denote it by  $mg'$ . The centrifugal force is directed along  $BA$ , and its magnitude is  $mp\Omega^2$ , where  $p = BA$ .

In order that the particle may remain in equilibrium relative to the earth, a third force must be applied to balance the gravitational force and the centrifugal force. This force must lie in the plane  $ABC$ , and its magnitude must be proportional to  $m$ . We denote it by  $mg$ ; it may be supplied by the tension in a string or plumb line, or by the reaction of a smooth plane. We define the *vertical*  $AV$  at  $A$  as the direction of this force and the *horizontal plane*  $HAH'$  as the plane perpendicular to it.

We now ask: As we range over the earth's surface, what is the relation between  $g$  and  $g'$ , and what is the inclination of the vertical to the direction of the gravitational force?

The *astronomical latitude*  $\lambda$  is defined as the elevation of the astronomical pole above the horizontal plane, i.e., the angle between  $SN$  and  $H'AH$ , or (equivalently) the angle between  $BA$  and  $AV$ . Let us denote by  $\theta$  the angle between  $CA$  and  $AV$ . Resolution of forces along and perpendicular to  $AC$  gives as conditions of equilibrium.

$$(5.318) \quad \begin{cases} g' = g \cos \theta + p\Omega^2 \cos (\lambda - \theta), \\ g \sin \theta = p\Omega^2 \sin (\lambda - \theta). \end{cases}$$

Now the ratio  $p\Omega^2/g$  is small; hence  $\theta$  is small, and  $\cos \theta$  differs from unity by a small quantity of the second order. To the first order of small quantities, we have

$$(5.319) \quad g' = g + p\Omega^2 \cos \lambda, \quad \theta = \frac{p\Omega^2}{g} \sin \lambda.$$

The terms involving  $\Omega$  are small, and to our order of approximation we are entitled to replace  $p$  and  $\lambda$  by approximate values. Now  $\lambda$  is approximately equal to the angle  $BAC$ , and so

$$p = CA \cos \lambda,$$

approximately. According to a well-known law of hydrostatics, the surface of the ocean must be tangent to the horizontal plane; in fact,  $AV$  is normal to the surface of our model of the earth. Thus, since  $\theta$  is small,  $CA$  is very nearly normal to this surface, and so  $CA = R$  approximately, where  $R$  is the radius of the earth in the first crude model. Hence (5.319) may be written

$$(5.320) \quad g' = g + R\Omega^2 \cos^2 \lambda,$$

$$(5.321) \quad \theta = \frac{R\Omega^2}{g} \sin \lambda \cos \lambda.$$

Thus we can find the gravitational intensity  $g'$  in terms of measurable quantities,  $g$  being measured by means of a pendulum (cf. Sec. 6.3). Theoretically,  $g$  is given by a measurement of the tension in a plumb line, but this is not a practical method. At the North and South Poles,  $g = 983 \text{ cm. sec}^{-2}$ ; at the Equator,  $g = 978 \text{ cm. sec}^{-2}$ . Equation (5.321) gives the deviation of the plumb line from the direction of the gravitational force; it is a maximum at a latitude of  $45^\circ$ .

Other effects of the earth's rotation will be treated in Sec. 13.5.

## 5.4. SUMMARY OF METHODS OF PLANE DYNAMICS

### I. Equations of motion of a particle.

$$(5.401) \quad m\mathbf{f} = \mathbf{P} \quad (\text{vector form});$$

$$(5.402) \quad m\ddot{x} = X, \quad m\ddot{y} = Y \quad (\text{Cartesian coordinates});$$

$$(5.403) \quad m\dot{q} = P_t, \quad \frac{mq^2}{\rho} = P_n \quad (\text{resolution along tangent and normal});$$

$$(5.404) \quad m(\ddot{r} - r\dot{\theta}^2) = R, \quad r^2\dot{\theta} = \text{const.} \quad (\text{central force}).$$

### II. Principle of angular momentum for a particle.

$$(5.405) \quad \dot{h} = N,$$

where

$$h = m(x\dot{y} - y\dot{x}) = mr^2\dot{\theta}.$$

### III. Principle of energy for a particle.

$$(5.406) \quad \dot{T} = \dot{W},$$

where

$$T = \frac{1}{2}m\dot{q}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad W = \text{work done};$$

$$(5.407) \quad T + V = E \quad (\text{conservation of energy}).$$

### IV. Principle of linear momentum for a system.

Coriolis force with components  $(2m\omega\dot{y}, -2m\omega\dot{x})$ ,

Centrifugal force with components  $(m\omega^2x, m\omega^2y)$ .

### EXERCISES V

1. At a certain instant, a particle of mass  $m$ , moving freely in a vertical plane under gravity, is at a height  $h$  above the ground and has a speed  $q$ . Use the principle of energy to find its speed when it strikes the ground.

2. What is the least number of revolutions per minute of a rotating drum, 2 feet in internal diameter, in order that a stone placed inside the drum may be carried right round? Assume that the contact between the stone and the drum is rough enough to prevent sliding. (The reaction of the drum on the stone must be directed inward.)

3. A skier, starting from rest, descends a slope 117 yards long and inclined at an angle of  $\sin^{-1} \frac{5}{13}$  to the horizontal. If the coefficient of friction between the skis and the snow is  $\frac{1}{6}$ , find his speed at the bottom of the slope. If the skier with his equipment weighs 200 lb., how much energy is dissipated in overcoming friction?

4. A bead of mass  $m$  slides on a smooth wire in the form of a parabola with axis vertical and vertex downward. If the bead starts from rest at an end of the latus rectum (of length  $4a$ ), find the speed with which it passes through the vertex. Find also the reaction of the wire on the bead at this point.

5. A heavy particle rests on top of a smooth fixed sphere. If it is slightly displaced, find the angular distance from the top at which it leaves the surface.

6. Two barges of masses  $m_1, m_2$  at a distance  $d$  from each other are connected by a cable of negligible weight. One barge is drawn up to the other by winding in the cable. If neither barge is anchored, find the distance through which each barge moves. (Neglect any frictional effects due to the water.)

7. An airplane with an air speed of 120 miles per hour starts from  $A$  to go to  $B$  which is northeast of  $A$ . If there is a wind blowing from the north at 20 miles per hour, in what direction must the pilot point the airplane if he wishes to go in a straight line from  $A$  to  $B$ ?

8. A heavy particle is suspended from a fixed point by a light string of length  $a$ . If the string would break under a tension equal to twice the weight of the particle, find the greatest angular velocity at which the string and particle can rotate as a conical pendulum without the string breaking.

9. A steamer sailing east at 24 knots is 1000 feet to the north of a launch which is proceeding north at 7 knots. Find the shortest subsequent distance between them if these courses are maintained. Draw rough diagrams, showing

# CHAPTER VI

## APPLICATIONS IN PLANE DYNAMICS—

### MOTION OF A PARTICLE

#### 6.1. PROJECTILES WITHOUT RESISTANCE

The science of *ballistics* is concerned with the motion of projectiles. The theory of the explosion of the charge and the motion of the projectile in the barrel of the gun belong to *interior* ballistics, with which we shall not be concerned. After the projectile leaves the barrel of the gun, it moves under the influence of gravity and the resistance of the air; the purpose of *exterior* ballistics is to predict, from given muzzle velocity and angle of elevation of the gun, the path or *trajectory* of the projectile.

On account of the complicated nature of the resistance of the air, an accurate mathematical prediction is not possible. The greatest difficulties arise from the fact that the projectile is of finite size. To avoid these, we regard the projectile as a particle.

In the present section, we shall make a further and much more drastic simplification; we shall assume that no resistance is offered by the air. We cannot claim that the theory based on this hypothesis gives results of much practical value in ballistics, except in the case of projectiles thrown with small velocities.\*

#### **The parabolic trajectory.**

Let  $Oxy$  be rectangular axes,  $Ox$  being horizontal and  $Oy$  vertical, directed upward. The equations of motion of a particle under the influence of gravity are

$$(6.101) \quad m\ddot{x} = 0, \quad m\ddot{y} = -mg.$$

Integration gives

$$(6.102) \quad \dot{x} = u_0, \quad \dot{y} = v_0 - gt,$$

$$(6.103) \quad x = x_0 + u_0t, \quad y = y_0 + v_0t - \frac{1}{2}gt^2,$$

where  $x_0, y_0, u_0, v_0$  are constants of integration. It is evident that  $(x_0, y_0)$  is the position and  $(u_0, v_0)$  the velocity, both at time  $t = 0$ . The equations (6.103) give the path, or trajectory, of the particle.

Obviously, at a certain instant ( $t = v_0/g$ ), we have  $\dot{y} = 0$ , so that at that instant the velocity is horizontal. Now the origin  $O$  and the instant from which  $t$  is measured may be chosen as we please. Let us choose  $O$  at the point where the velocity is horizontal and measure  $t$  from that instant. Then we have

$$(6.104) \quad \text{for } t = 0, \quad x = y = 0, \quad \dot{y} = 0,$$

Substituting in (6.102) and (6.103), we obtain

$$x_0 = y_0 = 0, \quad v_0 = 0,$$

and so the equations of the trajectory become

$$(6.105) \quad x = u_0 t, \quad y = -\frac{1}{2}gt^2.$$

Elimination of  $t$  gives the equation of the trajectory in the form

$$(6.106) \quad y = -\frac{gx^2}{2u_0^2},$$

a *parabola* with its vertex at the origin (Fig. 71).

The focus  $F$  of the parabola is situated at a distance  $a = \frac{1}{2}u_0^2/g$  below the vertex, and the directrix  $L$  is at the same height above the vertex. At any point on the trajectory the speed  $q$  is given by

$$(6.107) \quad \begin{aligned} q^2 &= \dot{x}^2 + \dot{y}^2 = u_0^2 + g^2 t^2 \\ &= u_0^2 - 2gy = 2g(a - y); \end{aligned}$$

thus *the speed at any point  $P$  on the trajectory is equal to the speed acquired in free fall to  $P$  from rest at the directrix  $L$ .*

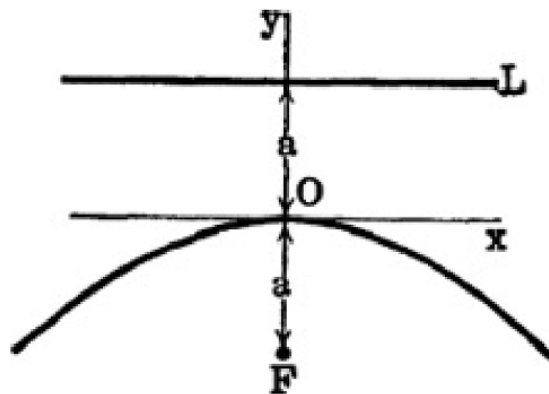


FIG. 71.—Parabolic trajectory, with focus  $F$ , vertex  $O$ , and directrix  $L$ .



From this it follows that, when a projectile is fired from a point  $P$  with speed  $q_0$ , the directrix of its parabolic trajectory is at the greatest height reached by a second projectile, fired straight up from  $P$  with the same speed  $q_0$ . In particular, we note that all trajectories obtained by firing projectiles at various inclinations in one vertical plane, but with a common initial speed  $q_0$ , have a geometrical property in common, namely, a common directrix.

The axes shown in Fig. 71 are the simplest for the discussion of general properties of the trajectory. But in ballistic problems it is preferable to take the origin at the point of projection and measure the time from the instant of firing (Fig. 72).

If  $\alpha$  is the inclination of the initial velocity to the horizontal, we have, as in (6.102) and (6.103),

$$(6.108) \quad \begin{cases} x = (q_0 \cos \alpha)t, \\ y = (q_0 \sin \alpha)t - \frac{1}{2}gt^2. \end{cases}$$

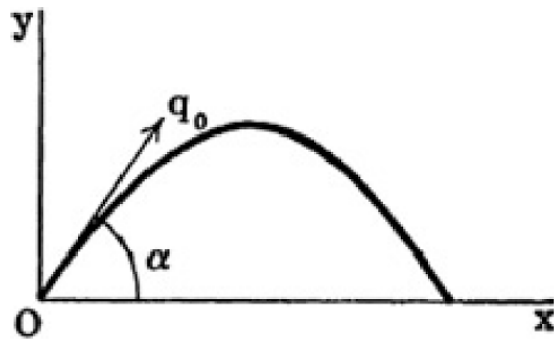


FIG. 72.—Parabolic trajectory referred to the point of projection.

The projectile strikes the ground when  $y = 0$ , that is, when

$$(6.109) \quad t = \frac{2q_0}{g} \sin \alpha;$$

then

$$(6.110) \quad x = \frac{q_0^2}{g} \sin 2\alpha.$$

This is the *range* of the projectile. We note that it is a maximum (for given  $q_0$ ) when  $\alpha = 45^\circ$ .

The greatest height attained by the projectile is obtained by putting  $\dot{y} = 0$ . Then, by (6.108),

(6.111)

$$t = \frac{q_0}{g} \sin \alpha, \quad y = \frac{q_0^2}{2g} \sin^2 \alpha.$$

### Limits of range.

Let us suppose that a gun gives to a projectile a muzzle velocity  $q_0$ . The gun can be pointed in any direction. What region in space can be reached by the projectile?

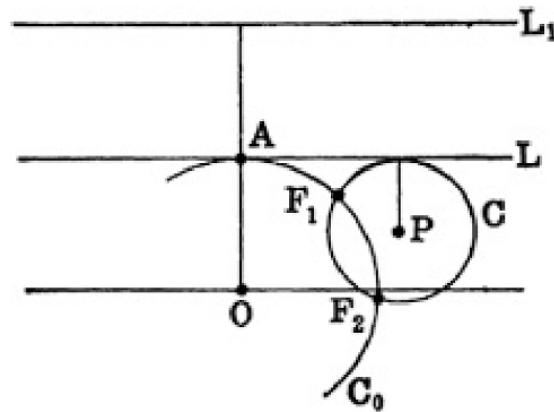


FIG. 73.—Construction for the foci of the two parabolic trajectories passing through  $P$ .

The question may be answered analytically, but the most elegant solution is geometrical.

We may confine our attention to one vertical plane through the gun, which is at  $O$  in Fig. 73. First we ask: Where is the focus of the trajectory passing through an assigned point  $P$ ? The answer is given by the following construction:

Draw the directrix  $L$ , at a height  $\frac{1}{2}q_0^2/g$  above  $O$ . Draw the circle  $C_0$  with center  $O$ , touching  $L$  at  $A$ . Since  $O$  is a point on the parabolic trajectory, the focus must lie at a distance  $OA$  from  $O$ , and so it must lie on  $C_0$ . Similarly, if the circle  $C$  is drawn with center  $P$  to touch  $L$ , the focus must lie on  $C$  also. Hence the focus must lie at an intersection of the circles  $C_0$  and  $C$ . In general, there will be either two points of intersection,  $F_1$  and  $F_2$ , or none. In the former case,  $P$  is within range and  $F_1, F_2$  are the foci of the *two* trajectories through it. In the latter case,  $P$  is out of range.

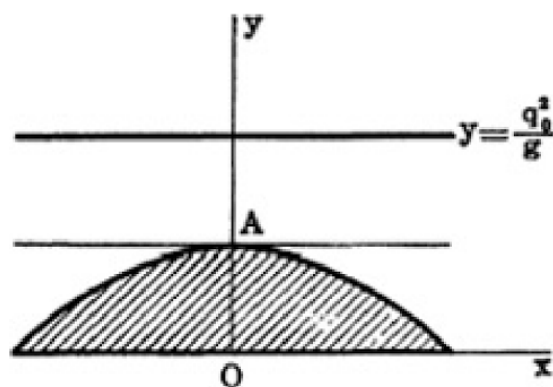


FIG. 74.—Paraboloidal region within range.

If  $P$  is at the limit of range, the circles  $C_0$  and  $C$  touch. Then  $P$  is equidistant from  $O$  and a horizontal line  $L_1$ , drawn at a height twice that of  $L$ , i.e., at a height  $q_0^2/g$ . Thus the locus of  $P$  in space is a paraboloid of revolution, having  $O$  for focus and  $A$  for vertex (Fig. 74). All points inside this paraboloid are within range, and all points outside it are out of range.

## 6.2. PROJECTILES WITH RESISTANCE

### General equations.

We turn now to the more practical problem in which the air exerts on the projectile (still regarded as a particle) a force  $R$  acting in a direction opposite to the velocity (Fig. 75).

In resolving forces and acceleration in order to obtain equations of motion in scalar form, we have a choice of two procedures: (i) resolution along horizontal and vertical directions, and (ii) resolution along the tangent and normal to the trajectory as in (5.103).

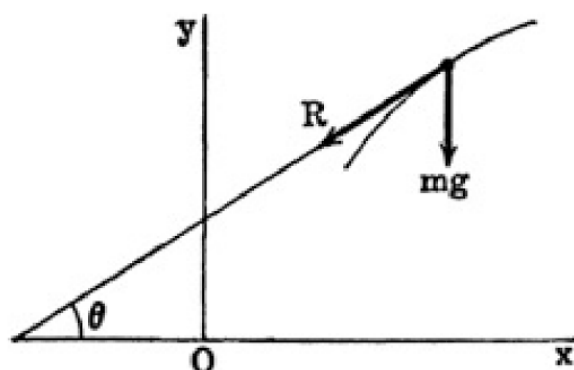


FIG. 75.—The forces acting on a projectile.

Denoting by  $\theta$  the inclination to the horizontal of the tangent to the trajectory, we obtain by (i) the equations

$$(6.201) \quad m\ddot{x} = -R \cos \theta, \quad m\ddot{y} = -R \sin \theta - mg.$$

Denoting by  $\rho$  the radius of curvature of the trajectory, we obtain by (ii) the equations

$$(6.202) \quad mq \frac{dq}{ds} = -R - mg \sin \theta, \quad \frac{mq^2}{\rho} = mg \cos \theta,$$

where  $q$  is the speed and  $ds$  an element of arc of the trajectory. The equations (6.202) are, of course, only a different mathematical expression of (6.201).

The resistance experienced by a given projectile depends on its speed and on the density of the air. Regarding the air as stratified into horizontal layers each of constant density, so that the density is a function of  $y$  only, we may express  $R$  in the form

$$(6.203) \quad R = R(y, q),$$

to show that it is a function of  $y$  and  $q$  only.

Since

$$(6.204) \quad \cos \theta = \frac{\dot{x}}{q}, \quad \sin \theta = \frac{\dot{y}}{q},$$

we may write (6.201) in the form

$$(6.205) \quad \ddot{x} = -\Phi \dot{x}, \quad \ddot{y} = -\Phi \dot{y} - g,$$

where  $\Phi$  is a function of  $y, q$ , namely,

$$(6.206) \quad \Phi(y, q) = \frac{R(y, q)}{mq}.$$

The mathematical problem of the determination of the trajectory is made much more difficult by the fact that there is no physically valid formula expressing  $R$  as a function of  $q$ . For small values of  $q$ ,  $R$  varies as  $q$ ; but this simple law breaks down before we reach those velocities which are of interest in ballistics. For low ballistic velocities,  $R$  varies as  $q^2$ ; but this law again breaks down when the velocity of the projectile approaches the velocity of sound. The law of dependence is then complicated and can be represented only graphically or by tables of values obtained experimentally. Hence, we must not expect to find any simple formulas for trajectories. In general, (6.205) must be integrated by a tedious process of step-by-step numerical integration. The differential equations are integrated approximately over small intervals of time; the errors due to approximation become insignificant when the intervals are very small.

These equations express  $x$ ,  $y$ ,  $t$  as functions of one parameter  $\theta$  and so determine the trajectory.

But we cannot use (6.211) until the function  $f(\phi)$  is known, i.e., until the differential equation (6.208) is integrated. If the function  $\phi(q)$  is general, the integration of (6.208) cannot even be reduced to quadratures. For some special forms of  $\phi(q)$  the integration can be reduced to quadratures, and in some cases the solution  $f(\theta)$  can be expressed in terms of elementary functions. We shall consider below the case  $\phi(q) = Cq^2$ , but before making this special choice of  $\phi$  we shall transform (6.208) by changing to a new independent variable  $\psi$ , defined by

$$\tanh \psi = \sin \theta.$$

It is easily seen that (6.208) transforms into

$$(6.212) \quad \frac{1}{q} \frac{dq}{d\psi} = \tanh \psi + \phi(q).$$

### Resistance varying as the square of the velocity.

Let us take the law of resistance to be

$$(6.213) \quad R = mg\phi(q), \quad \phi(q) = Cq^2,$$

where  $C$  is a constant. Division of (6.212) by  $-\frac{1}{2}q^2$  gives

$$(6.214) \quad \frac{d}{d\psi} \left( \frac{1}{q^2} \right) = -\frac{2}{q^2} \tanh \psi - 2C;$$

this is a standard type of equation, with solution

$$(6.215) \quad \begin{aligned} \frac{1}{q^2} &= \operatorname{sech}^2 \psi (A - 2C \int \cosh^2 \psi d\psi) \\ &= \cos^2 \theta [A - C \tanh^{-1} (\sin \theta)] - C \sin \theta, \end{aligned}$$

where  $A$  is a constant of integration, to be fixed by the initial conditions.

Theoretically at least, the equations (6.211) now determine the trajectory,  $f(\theta)$  being the reciprocal of the square root of the right-hand side of (6.215). But it is evident that the calculations involved are very complicated.

When the projectile moves in a vertical line, the problem is much simpler. If  $\mathbf{i}$  is a unit vector directed vertically upward and the position vector of the projectile is  $y\mathbf{i}$ , the acceleration is  $\ddot{y}\mathbf{i}$ . The force of gravity is  $-mg\mathbf{i}$ . The resistance is  $-mgC\dot{y}^2\mathbf{i}$  for motion upward ( $\dot{y} > 0$ ) and  $mgC\dot{y}^2\mathbf{i}$  for motion

downward ( $\dot{y} < 0$ ). Hence the equation of motion is

$$\begin{aligned} (6.216a) \quad \ddot{y} &= -gC\dot{y}^2 - g && \text{for motion upward;} \\ (6.216b) \quad \ddot{y} &= gC\dot{y}^2 - g && \text{for motion downward.} \end{aligned}$$

Since

$$\ddot{y} = \dot{y} \frac{d\dot{y}}{dy},$$

we have

$$(6.217a) \quad \frac{\dot{y} d\dot{y}}{1 + C\dot{y}^2} = -g dy \quad \text{for motion upward;}$$

$$(6.217b) \quad \frac{\dot{y} d\dot{y}}{1 - C\dot{y}^2} = -g dy \quad \text{for motion downward.}$$

Thus, on integration,

$$(6.218a) \quad y = -\frac{1}{2gC} \log (1 + C\dot{y}^2) + A \quad \text{for motion upward,}$$

$$(6.218b) \quad y = \frac{1}{2gC} \log (1 - C\dot{y}^2) + A' \quad \text{for motion downward,}$$

where  $A, A'$  are constants of integration.

Let us consider two special cases, corresponding to (a) a shell fired vertically upward, and (b) a bomb dropped vertically downward.

(a) Let  $q_0$  be the initial velocity of the shell, fired from  $y = 0$ . Then, by (6.218a), we have

$$(6.219) \quad y = \frac{1}{2gC} \log \frac{1 + Cq_0^2}{1 + C\dot{y}^2}.$$

The height  $h$  to which the shell rises is found by putting  $\dot{y} = 0$ ; thus,

$$(6.220) \quad h = \frac{1}{2gC} \log (1 + Cq_0^2).$$

If the resistance is small, this gives, approximately,

$$(6.221) \quad h = \frac{1}{2} \frac{q_0^2}{g} - \frac{1}{4} \frac{Cq_0^4}{g},$$

in which the first term is the well-known expression for height attained under no resistance.

(b) Let the bomb be dropped from  $y = 0$  with no velocity. Then (6.218b)



gives

$$(6.222) \quad y = \frac{1}{2gC} \log (1 - C\dot{y}^2),$$

$y$  being, of course, negative. When the bomb has dropped a distance  $h$ , we have

$$\log (1 - C\dot{y}^2) = -2gCh,$$

and hence its speed is

$$(6.223) \quad q = \sqrt{\frac{1 - e^{-2gCh}}{C}}.$$

As  $h$  tends to infinity,  $q$  tends to  $C^{-\frac{1}{2}}$ . This is the *limiting velocity*; its value is

$$(6.224) \quad C^{-\frac{1}{2}} = q \sqrt{mg/R},$$

where  $R$  is the resistance at any speed  $q$ .

Relations connecting  $y$  and  $t$  may be obtained from (6.216) by integration. Thus, (6.216a) may be written

$$\frac{d\dot{y}}{1 + C\dot{y}^2} = -g dt.$$

One integration gives  $\dot{y}$  in terms of  $t$ , and a second integration gives  $y$ .

### Resistance varying directly as the velocity.

Although the law of resistance

$$(6.225) \quad R = mgCq$$

is not accurate physically, it is so simple to treat mathematically that it is a useful approximation—at least an improvement over the assumption of no resistance at all.

Turning back to (6.206), we note that  $\Phi$  is now a constant ( $\Phi = gC$ ), and the equations (6.205) are easy to handle, because the variables  $x$  and  $y$  are separated. We obtain, on integration,

$$(6.226) \quad \begin{cases} x = x_0 + \frac{u_0}{\Phi} (1 - e^{-\Phi t}), \\ y = y_0 - \frac{gt}{\Phi} + \left( \frac{v_0}{\Phi} + \frac{g}{\Phi^2} \right) (1 - e^{-\Phi t}), \end{cases}$$

where  $x_0, y_0$  are the coordinates and  $u_0, v_0$  the components of velocity for  $t = 0$ .

When  $\Phi$  is small, these equations yield approximately

$$(6.227) \quad \begin{cases} x = x_0 + u_0 t - \frac{1}{2} \Phi u_0 t^2, \\ y = y_0 + v_0 t - \frac{1}{2} g t^2 - \frac{1}{2} \Phi v_0 t^2 \left( 1 - \frac{1}{8} \frac{g t}{v_0} \right), \end{cases}$$

in which the terms independent of  $\Phi$  correspond to the parabolic trajectory.

### 6.3. HARMONIC OSCILLATORS

#### The simple pendulum.

A simple pendulum consists of a heavy particle attached to one end of a light rod or inextensible string, the other end of the rod or string being attached to a fixed point. We consider only motions of the pendulum in which the string remains in a definite vertical plane. In Fig. 76,  $B$  is the point of attachment and  $A$  is the particle (of mass  $m$ ), drawn aside from its position of equilibrium  $O$ .  $Oxy$  are rectangular axes in the plane of motion,  $Ox$  being horizontal.

The particle moves under the influence of two forces: (i) its weight  $mg$ , and (ii) the tension  $S$  in the string. Since  $S$  acts along the normal to the circular path of the particle, it is clear that the dynamical problem presented by the simple pendulum is precisely the same as that of the motion of a particle on a smooth circular supporting curve, fixed in a vertical plane.

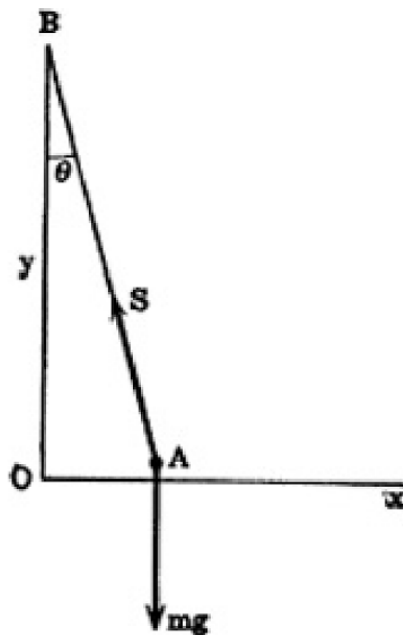


FIG. 76.—The simple pendulum.

Let  $AB = l$ ,  $\angle BOA = \theta$ ; then

$$(6.301) \quad \cos \theta = \frac{l - y}{l}, \quad \sin \theta = \frac{x}{l},$$

where  $x, y$  are the coordinates of  $A$ . The equations of motion are

$$(6.302) \quad m\ddot{x} = -S \sin \theta, \quad m\ddot{y} = S \cos \theta - mg.$$

Let us investigate small oscillations about the position of equilibrium, assuming  $x$  and its derivatives to be small. Then  $y$  and its derivatives are small, of the second order, and  $\cos \theta$  differs from unity by a small quantity of the second order. Thus, to the first order inclusive, the second of (6.302) gives

$$(6.303) \quad S = mg,$$

and substitution from this and from (6.301) in (6.302) gives

$$(6.304) \quad \ddot{x} + p^2 x = 0, \quad p = \sqrt{\frac{g}{l}}.$$

The general solution of this differential equation is

$$(6.305) \quad x = A \cos pt + B \sin pt,$$

where  $A, B$  are constants, to be determined by the initial conditions. A motion given by an equation of this form is called *simple harmonic*.

Differentiation of (6.305) gives

$$(6.306) \quad \dot{x} = -Ap \sin pt + Bp \cos pt.$$

We note that  $x$  and  $\dot{x}$  have the same pair of values at times

$$t_1, t_1 + \tau, t_1 + 2\tau, t_1 + 3\tau, \dots$$

where  $t_1$  is arbitrary, and

$$(6.307) \quad \tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{l}{g}}.$$

When the position and velocity of a particle are repeated over and over again at equal intervals of time, we say that the motion is *periodic*; the interval is called its *periodic time*. Hence the above motion of a simple pendulum is periodic, with periodic time given by (6.307).

In Sec. 13.2 the finite oscillations of a pendulum will be discussed, and it will be found that the finite oscillations are periodic but not simple harmonic; the formula for the periodic time is different.

## The harmonic oscillator.

The simple pendulum, executing small oscillations, is only one physical example of a type of dynamical system of frequent occurrence. Many problems of oscillation can be discussed in a single mathematical form; so we create a single mathematical model for them all. This model is called the *harmonic oscillator* (Fig. 77).

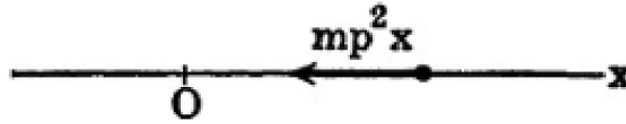


FIG. 77.—The harmonic oscillator.

The harmonic oscillator consists of a particle which can move on a straight line, which we shall take for  $x$ -axis. It is attracted toward the origin by a controlling force varying as the distance. If  $\mathbf{i}$  is a unit vector in the positive direction of the  $x$ -axis, the controlling force may be written  $-mp^2x\mathbf{i}$  where  $m$  is the mass of the particle and  $p$  is a constant. The acceleration is  $\ddot{x}\mathbf{i}$ , and hence the equation of motion is

$$(6.308) \quad m\ddot{x}\mathbf{i} = -mp^2x\mathbf{i},$$

or, in scalar form,

$$(6.309) \quad \ddot{x} + p^2x = 0.$$

The general solution of this equation may be written in the form (6.305), that is,

$$(6.310) \quad x = A \cos pt + B \sin pt,$$

so that the motion is simple harmonic.

Let us now define constants  $a, \epsilon$  by the equations

$$(6.311) \quad a = \sqrt{A^2 + B^2}, \quad \cos \epsilon = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \epsilon = -\frac{B}{\sqrt{A^2 + B^2}}.$$

Then (6.310) may be written

$$(6.312) \quad x = a \cos (pt + \epsilon).$$

We observe that  $x$  covers the range  $(-a, a)$  and that the motion is periodic with periodic time  $2\pi/p$ . The following terminology is used in connection with the harmonic oscillator:

amplitude =  $a$ ,

period or periodic time =  $\tau = 2\pi/p$ ,

frequency = number of oscillations per unit time =  $1/\tau = p/2\pi$ ,

phase =  $pt + \epsilon$

### Effect of a disturbing force.

Let us now suppose that, in addition to the controlling force, there acts on the harmonic oscillator a force whose component in the positive direction of the  $x$ -axis is  $mX$ . The equation of motion (6.308) is now modified to the form

$$(6.313) \quad m\ddot{x} = -mp^2x + mX,$$

or, in scalar form,

$$(6.314) \quad \ddot{x} + p^2x = X.$$

First let us suppose that  $X$  is constant. The general solution of (6.314) is then

$$(6.315) \quad x = \frac{X}{p^2} + a \cos (pt + \epsilon).$$

This motion is simple harmonic, but the center of the oscillations is displaced to the position  $x = X/p^2$ , as is seen by writing (6.315) in the form

$$(6.316) \quad x - \frac{X}{p^2} = a \cos (pt + \epsilon).$$

When the oscillation or vibration has a high frequency, so that  $p$  is large, the displacement of the center is small.

Let us now suppose that  $X$  is itself simple harmonic, varying according to the formula

$$(6.317) \quad X = k \cos ct,$$

where  $k, c$  are constants. Then the equation of motion (6.314) reads

$$(6.318) \quad \ddot{x} + p^2x = k \cos ct.$$

In the general case where  $c \neq p$ , the solution is

$$(6.319) \quad x = a \cos (pt + \epsilon) + \frac{k}{p^2 - c^2} \cos ct,$$

where  $a, \epsilon$  are constants, to be determined by the initial conditions. Thus the motion is a superposition of an undisturbed simple harmonic motion and a second motion; the latter has the same period as the force, and an amplitude which becomes large when the difference between the periods of the free oscillator and the disturbing force becomes small. The great increase in the amplitude of the oscillations under this last condition is called *resonance*. We shall discuss it again below, taking resistance into consideration.

### Damped oscillations.

Let us now suppose that, in addition to the controlling force, there acts on the particle of a harmonic oscillator a force of resistance proportional to the velocity, called the *damping force*. The component of this force in the positive direction of the  $x$ -axis may be written  $-2m\mu\dot{x}$ , where  $\mu$  is a positive constant. The equation of motion (6.308) is modified to

$$(6.320) \quad m\ddot{x} = -mp^2x - 2m\mu\dot{x},$$

or, in scalar form,

$$(6.321) \quad \ddot{x} + 2\mu\dot{x} + p^2x = 0.$$

Now,

$$(6.322) \quad x = Ce^{nt}$$

will satisfy this equation provided that  $n$  satisfies the *characteristic equation*

$$(6.323) \quad n^2 + 2\mu n + p^2 = 0.$$

This equation has two roots, which may be real or complex; they are

$$(6.324) \quad n_1, n_2 = -\mu \pm \sqrt{\mu^2 - p^2}.$$

Let us define a real positive number  $l$  by

$$(6.325) \quad l = \sqrt{|\mu^2 - p^2|},$$

the sign  $||$  indicating absolute value. We have then two cases to discuss: (i) light damping,  $\mu < p$ ; (ii) heavy damping,  $\mu > p$ .

CASE (i) *Light damping* ( $\mu < p$ ).

In this case,

$$(6.326) \quad n_1, n_2 = -\mu \pm il,$$



and the general solution of (6.321) is

$$(6.327) \quad x = C_1 e^{(-\mu + i\omega)t} + C_2 e^{(-\mu - i\omega)t}.$$

This may be written

$$(6.328) \quad x = e^{-\mu t}(A \cos \omega t + B \sin \omega t),$$

where

$$(6.329) \quad A = C_1 + C_2, \quad B = i(C_1 - C_2).$$

Since, for physical reasons, we are interested only in real values of  $x$ , it is evident that, although the differential equation is satisfied by (6.327) with  $C_1, C_2$  complex constants, we should adopt as our final solution (6.328) with *real* constants  $A, B$ . These may be chosen to satisfy the initial conditions, i.e., to give assigned values to  $x$  and  $\dot{x}$  for  $t = 0$ .

The motion given by (6.328) may also be written

$$(6.330) \quad x = a e^{-\mu t} \cos(\omega t + \epsilon),$$

where  $a, \epsilon$  are constants chosen to fit the initial conditions. This is not a simple harmonic motion, not being of the form (6.312). The factor  $e^{-\mu t}$  indicates a general decay of the oscillations,  $x$  tending to zero as  $t$  tends to infinity.

By changing the instant from which  $t$  is measured, we can get rid of  $\epsilon$  in (6.330), obtaining the simpler expression

$$(6.331) \quad x = a e^{-\mu t} \cos \omega t.$$

On plotting  $x$  as a function of  $t$ , we obtain the graph shown in Fig. 78. (The graph of the simple harmonic motion

$$x = a \cos \omega t$$

is a cosine curve, resembling the above curve, except for the latter's tendency to die away.) The curve of (6.331) should be compared with the curves

$$(6.332a) \quad x = a e^{-\mu t},$$

$$(6.332b) \quad x = -a e^{-\mu t},$$

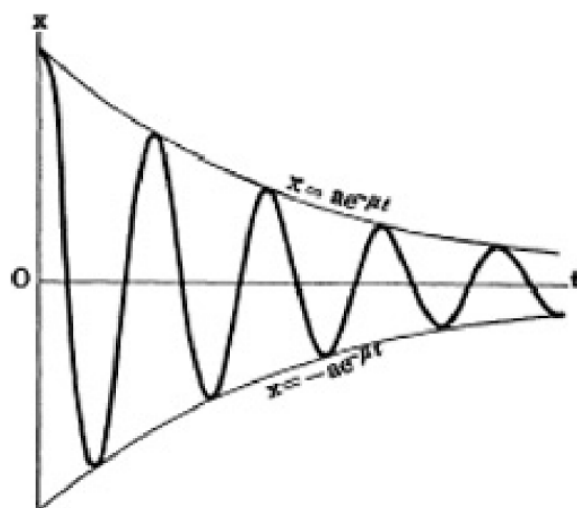


FIG. 78.—Position-time graph for a lightly damped harmonic oscillator.

also shown in Fig. 78. It is easily seen that (6.331) touches (6.332a) and (6.332b) at

$$(6.333a) \quad t = \frac{2n\pi}{l},$$

$$(6.333b) \quad t = \frac{(2n+1)\pi}{l},$$

respectively,  $n$  being any integer.

We may regard (6.331) as a harmonic motion with decaying amplitude, given by  $ae^{-\mu t}$ . But the following is a more accurate description.

The maxima and minima of (6.331) occur at instants  $t = t_n$ , where

$$(6.334) \quad t_n = n\pi - \alpha, \quad \tan \alpha = \frac{\mu}{l},$$

$n$  being any integer. The common interval is

$$(6.335) \quad \tau = \frac{\pi}{l};$$

thus  $2\pi/l$  may be referred to as the *period* of the oscillations. The values of  $x$  corresponding to (6.334) are

$$(6.336) \quad x_n = (-1)^n ae^{-\mu t_n} \cos \alpha.$$

Thus successive values are connected by

$$(6.337) \quad \frac{x_{n+1}}{x_n} = -e^{-\mu(t_{n+1}-t_n)} = -e^{-\pi\mu/l}.$$

The ratio of successive swings to opposite sides is, in absolute value,

$$(6.338) \quad \left| \frac{x_{n+1}}{x_n} \right| = e^{-\pi\mu/l}.$$

It is usual to define the *logarithmic decrement* of the damped oscillation as the logarithm of the reciprocal of this ratio to base 10; its value is  $(\pi\mu/l) \log_{10} e$ .

CASE (ii) *Heavy damping* ( $\mu > p$ ).

In the case of heavy damping, we have, by (6.324),

$$(6.339) \quad n_1, n_2 = -\mu \pm l,$$

and so the general solution of (6.321) is

$$(6.340) \quad x = e^{-\mu t}(Ae^{lt} + Be^{-lt}).$$

Then,

$$(6.341) \quad \dot{x} = e^{-\mu t}[A(l - \mu)e^{lt} - B(l + \mu)e^{-lt}].$$

This vanishes only if

$$(6.342) \quad e^{2lt} = \frac{B(l + \mu)}{A(l - \mu)}.$$

Since  $e^{2lt}$  is a steadily increasing function of  $t$ , there can be at most one solution. Hence the velocity of the oscillator vanishes at one instant at most. The motion is non-oscillatory, or *deadbeat*,  $x$  tending to zero as  $t$  tends to infinity, since  $\mu > l$ .

## Forced oscillations.

Finally, let us suppose that a harmonic oscillator is subject to

- (i) a controlling force ( $-mp^2x$ ),
- (ii) a damping force ( $-2m\mu\dot{x}$ ),
- (iii) a disturbing force ( $mk \cos ct$ ).

The equation of motion is now

$$(6.343) \quad x + 2\mu\dot{x} + p^2x = k \cos ct.$$

The general solution is

$$(6.344) \quad x = x_1 + x_2,$$

where  $x_1$  satisfies

$$(6.345) \quad x_1 + 2\mu\dot{x}_1 + p^2x_1 = 0,$$

and contains two constants of integration, while  $x_2$  is any particular solution of

$$(6.346) \quad x_2 + 2\mu\dot{x}_2 + p^2x_2 = k \cos ct.$$

Then  $x_1$  corresponds to the motion of the damped oscillator without disturbing force; it is given by (6.330) or (6.340), according as the damping is light or heavy.

As for  $x_2$ , (6.346) is satisfied by the expression

$$(6.347) \quad x_2 = E \cos ct + F \sin ct,$$

where

$$(6.348) \quad E = \frac{k(p^2 - c^2)}{(p^2 - c^2)^2 + (2\mu c)^2}, \quad F = \frac{2\mu ck}{(p^2 - c^2)^2 + (2\mu c)^2}.$$

This solution is most easily found by replacing  $\cos ct$  by  $e^{ict}$  in (6.346) and finding a complex constant  $G$  so that  $Ge^{ict}$  is a solution;  $x_2$  will then be the real part of this solution. Alternatively, we may substitute (6.347) directly in (6.346) to find the constants.

If we define

$$(6.349) \quad b = \frac{k}{\sqrt{(p^2 - c^2)^2 + (2\mu c)^2}},$$

$$\cos \eta = \frac{p^2 - c^2}{\sqrt{(p^2 - c^2)^2 + (2\mu c)^2}}, \quad \sin \eta = \frac{-2\mu c}{\sqrt{(p^2 - c^2)^2 + (2\mu c)^2}},$$

we may write (6.347) in the form

$$(6.350) \quad x_2 = b \cos (ct + \eta).$$

Thus the general motion of the damped oscillator under the influence of the disturbing force is

$$(6.351) \quad x = x_1 + b \cos (ct + \eta).$$

As  $t \rightarrow \infty$ ,  $x_1 \rightarrow 0$ ; thus after a long time the motion tends to the *forced oscillation*

$$(6.352) \quad x = b \cos (ct + \eta).$$

This is a simple harmonic motion of amplitude  $b$  and period equal to that of the disturbing force, but there is a difference in phase.

If the period of the disturbing force is equal to the free period of the oscillator, we have the case of *resonance*. Then  $c = p$ ,  $\eta = -\frac{1}{2}\pi$ , and so

$$(6.353) \quad x = \frac{k}{2\mu p} \cos (pt - \tfrac{1}{2}\pi) = \frac{k}{2\mu p} \sin pt,$$

the disturbing force being  $mk \cos pt$ . The difference of  $\frac{1}{2}\pi$  in phase is interesting.

#### 6.4. GENERAL MOTION UNDER A CENTRAL FORCE

##### Cartesian equations and the law of direct distance.

Consider a particle of mass  $m$  attracted toward a fixed point  $O$  by a force  $mP$ . (We may include the case of repulsion by taking  $P$  negative.) For rectangular Cartesian coordinates  $Oxy$ , the equations of motion are

$$(6.401) \quad m\ddot{x} = -\frac{mPx}{r}, \quad m\ddot{y} = -\frac{mPy}{r},$$

where  $r^2 = x^2 + y^2$ . We refer to this as motion under a *central force*, because the line of action of the force passes through a fixed center  $O$ .

As an application of (6.401), let us consider the *law of direct distance*, meaning thereby that  $P$  is proportional to  $r$ . Let us confine our attention to an attractive force, putting

$$(6.402) \quad P = k^2r,$$

where  $k$  is a constant.

Then (6.401) read

$$(6.403) \quad \ddot{x} + k^2x = 0, \quad \ddot{y} + k^2y = 0;$$

these equations have the general solutions

$$(6.404) \quad \begin{cases} x = A \cos kt + B \sin kt, \\ y = C \cos kt + D \sin kt, \end{cases}$$

where the coefficients are constants, to be fixed by the initial conditions.

If we solve (6.404) for  $\cos kt$  and  $\sin kt$ , and eliminate  $t$  by the identity

$$\cos^2 kt + \sin^2 kt = 1,$$

we get

$$(6.405) \quad (Cx - Ay)^2 + (Dx - By)^2 = (BC - AD)^2.$$

This is a central conic and necessarily an ellipse since  $x, y$  remain finite, as we

see from (6.404). Thus *the orbit described under a central attractive force varying directly as the distance is an ellipse having its center at the center of force*. This motion is called *elliptic harmonic*.

To illustrate the significance of the constants in (6.404), let us suppose that at time  $t = 0$  the particle is at  $x = a$ ,  $y = 0$ , moving with velocity  $v_0$  in the direction of the  $y$ -axis, so that  $\dot{x} = 0$ ,  $\dot{y} = v_0$ . Putting this information into the equations (6.404), first as they stand and then in the form obtained by differentiation, we get

$$a = A, 0 = C,$$

$$0 = B, v_0 = Dk,$$

and so the motion is given by

$$(6.406) \quad x = a \cos kt, \quad y = \frac{v_0}{k} \sin kt.$$

## Polar coordinates.

Returning to the general problem of motion under a central force, we shall now obtain equations easier to handle than (6.401). It is only in the case of the law of direct distance that (6.401) are convenient.

Since the force on the particle passes through  $O$ , it has no moment about  $O$ . Hence, by the principle of angular momentum (5.214), the angular momentum about  $O$  is constant. We shall change slightly the notation of [Chap. V](#), now letting  $h$  denote angular momentum per unit mass; then, by (5.106),

$$(6.407) \quad h = r^2\dot{\theta} = \text{constant},$$

where  $r$ ,  $\theta$  are the polar coordinates of the particle. An alternative expression for  $h$  is  $pq$ , where  $p$  is the perpendicular from  $O$  on the velocity vector  $q$ .

If we follow the radius vector, drawn from  $O$  to the particle, we observe that when it turns through an infinitesimal angle  $d\theta$  it sweeps out an area  $\frac{1}{2}r^2 d\theta$ . Thus the *areal velocity*  $\dot{A}$  may be defined as

$$(6.408) \quad \dot{A} = \frac{1}{2}r^2\dot{\theta},$$

$A$  being, in fact, the total area swept out from some initial instant. We observe that  $h$  is twice the areal velocity, and by (6.407) we have the following important result: *In motion under a central force, the areal velocity is constant.*

This fact is used to simplify the problem of determining the orbit. We define



$$(6.409) \quad u = \frac{1}{r},$$

the reciprocal of the radius vector. Then, since (6.407) may be written

$$(6.410) \quad \dot{\theta} = hu^2,$$

we have

$$(6.411) \quad \begin{cases} \dot{r} = -\frac{1}{u^2} \frac{du}{d\theta} \dot{\theta} = -h \frac{du}{d\theta}, \\ \ddot{r} = -h \frac{d^2u}{d\theta^2} \dot{\theta} = -h^2 u^2 \frac{d^2u}{d\theta^2}, \\ \ddot{r} - r\dot{\theta}^2 = -h^2 u^2 \left( \frac{d^2u}{d\theta^2} + u \right), \\ \dot{r}^2 + r^2 \dot{\theta}^2 = h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right]. \end{cases}$$

The vector equation of motion is

$$(6.412) \quad m\mathbf{f} = m\mathbf{P},$$

where  $\mathbf{P}$  is the attractive force per unit mass. Resolution along the radius vector gives, by (4.107),

$$(6.413) \quad \ddot{r} - r\dot{\theta}^2 = -P,$$

where  $P$  is the inward component of  $\mathbf{P}$ . By (6.411), we obtain

$$(6.414) \quad \frac{d^2u}{d\theta^2} + u = \frac{P}{h^2 u^2}.$$

*This is the differential equation of the orbit of a particle moving under an attractive central force  $P$  per unit mass. If we can solve this equation, obtaining  $u$  as a function of  $\theta$ , we have the equation of the orbit in polar coordinates.*

Henceforth, let us suppose that  $P$  is a function of  $r$  only. It is easy to see that the work done in passing from one position to another is then independent of the path described. Thus the system is conservative, and we may use the principle of energy. Let  $T$ ,  $V$ ,  $E$  be respectively the kinetic energy, potential energy, and constant total energy, all per unit mass. Then

$$(6.415) \quad T + V = E.$$

Now, by (6.411),

$$(6.416) \quad T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right].$$

The potential energy per unit mass is such that

$$\mathbf{P} = -\text{grad } V,$$

so that, since  $P$  is the *inward* component of  $\mathbf{P}$ ,

$$(6.417) \quad P = \frac{dV}{dr}, \quad V = \int_{r_0}^r P \, dr,$$

where  $r_0$  is some constant. Hence, (6.415) gives

$$(6.418) \quad \left( \frac{du}{d\theta} \right)^2 + u^2 = \frac{2(E - V)}{h^2}.$$

This equation is really equivalent to (6.414), as may be seen on differentiation. We are, of course, to remember that  $V$ , being a function of  $r$ , is also a function of  $u$ .

### Apsides and apsidal angles.

An *apse* is a point on an orbit at a maximum or minimum distance from the center of force. The condition for an apse is  $\dot{r} = 0$ , or, equivalently,

$$(6.419) \quad \frac{du}{d\theta} = 0.$$

From (6.418) it follows that, at an apse,

$$(6.420) \quad u^2 = \frac{2(E - V)}{h^2}.$$

Since  $V$  is a function of  $u$ , this is an equation to determine the values of  $u$  at the apsides, supposing the constants  $E$  and  $h$  known.

As an illustration, let us return to the case  $P = k^2r$ . Then

$$(6.421) \quad V = \frac{1}{2}k^2r^2 = \frac{1}{2} \frac{k^2}{u^2},$$

and so (6.420) may be written

$$(6.422) \quad u^4 - \frac{2}{h^2} (Eu^2 - \frac{1}{2}k^2) = 0.$$

This quadratic equation in  $u^2$  yields roots  $u_1^2$ ,  $u_2^2$ , which will be the squares of the reciprocals of the semiaxes of the elliptical orbit.

By a study of apsides, we may obtain a general description of an orbit without actually solving the differential equation (6.414). To establish an important feature, let us temporarily forget the dynamical problem and think of a differential equation

$$(6.423) \quad \frac{d^2y}{dx^2} = f(y),$$

to be solved under the initial conditions  $y = y_0$ ,  $dy/dx = 0$  for  $x = 0$ . These initial conditions determine a unique solution. Since the transformation  $x = -x'$  leaves the form of the differential equation and the initial conditions unaltered, it is evident that the curve  $y = F(x)$ , which satisfies (6.423) and the initial conditions, is symmetric with respect to the  $y$ -axis.

Since  $P$  is a function of  $u$ , it is clear that (6.414) and (6.423) are equations of the same form, with the correspondence  $u \rightarrow y$ ,  $\theta \rightarrow x$ . If we measure  $\theta$  from an apse, the initial conditions for (6.414) are the same as those for (6.423). Hence, *a central orbit is symmetric with respect to the line drawn from the force center to an apse.*

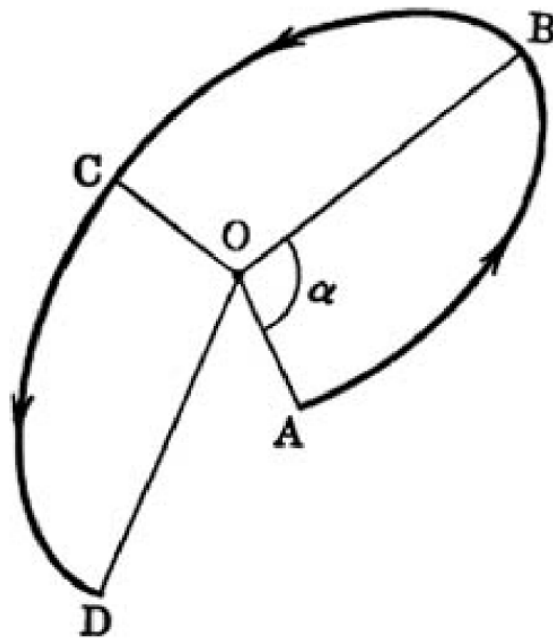


FIG. 79.—Symmetry of a central orbit with respect to an apse line.

This result throws much light on the general structure of a central orbit. Let  $A$  and  $B$  (Fig. 79) be consecutive apsides, and let us suppose that the portion

$AB$  of the orbit is known. The orbit is symmetrical about  $OB$ ; hence, we may obtain some more of the orbit by folding the portion  $AB$  over the line  $OB$ , obtaining  $BC$ , with an apse at  $C$  by symmetry. The apsidal distance  $OC$  is equal to the apsidal distance  $OA$ . Again, folding  $BC$  over  $OC$ , we get  $CD$  with an apse at  $D$ , and  $OD = OB$ .

The following facts are now clear:

(i) Any central orbit has only two apsidal distances. The orbit is a curve touching two concentric circles, the radii of which are the two apsidal distances.

(ii) Once the orbit between two consecutive apsides is known, the whole of the orbit may be constructed by operations of folding over apsidal radii.

(iii) The angle subtended at the center by the arc joining consecutive apsides is a constant. It is called the *apsidal angle*.

In some cases, (i) may be violated. The radius of the inner circle may be zero, or that of the outer circle may be infinite. But these are to be regarded as exceptional cases.

Let us now consider how the apsidal angle is to be found. When  $u$  is increasing, (6.418) gives

$$(6.424) \quad \frac{du}{d\theta} = \sqrt{F(u)},$$

where

$$(6.425) \quad F(u) = \frac{2(E - V)}{h^2} - u^2.$$

Thus

$$d\theta = \frac{du}{\sqrt{F(u)}};$$

and so, if  $u_1, u_2$  are the reciprocals of the apsidal distances (with  $u_1 < u_2$ ), the apsidal angle  $\alpha$  is

$$(6.426) \quad \alpha = \int_{u=u_1}^{u=u_2} d\theta = \int_{u_1}^{u_2} \frac{du}{\sqrt{F(u)}}.$$

As an illustration, let us consider the law of direct distance, where  $V$  is given by (6.421). Here  $F(u)$  is of the form

$$F(u) = \frac{1}{u^2} (A + Bu^2 - u^4),$$

where  $A$  and  $B$  are constants. But  $F(u) = 0$  at an apse; hence  $u = u_1$ ,  $u = u_2$  are roots of  $F(u) = 0$ , and so

$$F(u) = \frac{(u_2^2 - u^2)(u^2 - u_1^2)}{u^2}.$$

Thus, by (6.426),

$$(6.427) \quad \alpha = \int_{u_1}^{u_2} \frac{u \, du}{\sqrt{(u_2^2 - u^2)(u^2 - u_1^2)}} = \frac{1}{2}\pi,$$

as we already knew from the fact that the orbit is a central ellipse.

### Stability of circular orbits.

In a circular orbit  $u$  is constant. Hence by (6.414) the possible radii of circular orbits are determined by

$$(6.428) \quad u = \frac{P(u)}{h^2 u^2}.$$

With an attractive force ( $P > 0$ ), we can obtain a circular orbit of any radius, by projecting the particle at right angles to the radius vector with that velocity which makes

$$(6.429) \quad h^2 = \frac{P(u)}{u^3}.$$

But the question arises: Are these circular orbits *stable* or *unstable*? In other words, if slightly disturbed, will the resulting orbit lie close to the original circular orbit, or will it deviate far from it? This question is important physically, because in nature small disturbances are always present, and they will destroy an unstable circular orbit. The only circular orbits we can hope to observe are those that are stable.

It should be clearly understood that we shall not consider the effects of disturbing forces which *continue* to act on the particle; we assume that the position and velocity of the particle have been disturbed and investigate the resulting motion under the original central force.

Let  $u = u_0$  and  $h = h_0$  in the circular orbit. Then, by (6.429),

$$(6.430) \quad h_0^2 = \frac{P(u_0)}{u_0^3}.$$

To study the disturbance, we put

$$(6.431) \quad u = u_0 + \xi,$$

where  $\xi$  and its derivatives are assumed to be small. We also assume that  $h - h_0$  is small. Substitution in (6.414) gives

$$(6.432) \quad \frac{d^2\xi}{d\theta^2} + u_0 + \xi = \frac{P(u_0 + \xi)}{h^2(u_0 + \xi)^2}.$$

Now, on expansion in powers of  $\xi$ ,

$$(6.433) \quad \frac{P(u_0 + \xi)}{h^2(u_0 + \xi)^2} = \frac{1}{h^2 u_0^2} \left(1 + \frac{\xi}{u_0}\right)^{-2} (P_0 + \xi P'_0 + \dots) \\ = \frac{P_0}{h^2 u_0^2} \left[1 + \xi \left(\frac{P'_0}{P_0} - \frac{2}{u_0}\right) + \dots\right],$$

where  $P' = dP/du$ , and the subscript 0 indicates evaluation for  $u = u_0$ .

Then, to the first order in small quantities, (6.432) becomes

$$(6.434) \quad \frac{d^2\xi}{d\theta^2} + A\xi = B,$$

where

$$(6.435) \quad A = 1 - \frac{P_0}{h_0^2 u_0^2} \left(\frac{P'_0}{P_0} - \frac{2}{u_0}\right) = 3 - \frac{u_0 P'_0}{P_0},$$

by (6.430);  $B$  is another constant whose value does not interest us. The solution of (6.434) is

$$(6.436a) \quad \xi = \frac{B}{A} + C_1 \cos(\sqrt{A} \cdot \theta) + C_2 \sin(\sqrt{A} \cdot \theta),$$

$$(6.436b) \quad \xi = \frac{B}{A} + C_1 \cosh(\sqrt{-A} \cdot \theta) + C_2 \sinh(\sqrt{-A} \cdot \theta),$$

$$(6.436c) \quad \xi = \frac{1}{2}B\theta^2 + C_1\theta + C_2,$$

according as  $A > 0$ ,  $A < 0$ , or  $A = 0$ .

Of these solutions, only (6.436a) remains permanently small; the others



increase indefinitely with  $\theta$ . Hence *the circular orbit of radius  $1/u_0$  is stable if, and only if,*

$$(6.437) \quad \frac{u_0 P'_0}{P_0} < 3.$$

In particular, let us consider the case of a force varying inversely as the  $n$ th power of the distance so that

$$(6.438) \quad P = \frac{k}{r^n} = ku^n.$$

Then

$$\frac{uP'}{P} = n,$$

and so *circular orbits under the attractive force (6.438) are stable if, and only if,*

$$(6.439) \quad n < 3.$$

Thus the law of direct distance ( $n = -1$ ) and the law of the inverse square ( $n = 2$ ) give stable circular orbits; the law of the inverse cube ( $n = 3$ ) gives unstable circular orbits.

## 6.5. PLANETARY ORBITS

### The law of the inverse square.

As already remarked in Sec. 3.1, Newton's law of gravitational attraction states that two particles of masses  $m, m'$ , at a distance  $r$  apart, attract one another with equal and opposite forces of magnitude

$$(6.501) \quad \frac{Gmm'}{r^2},$$

where  $G$  is the gravitational constant.

Coulomb's law of electrostatic attraction states that two particles carrying electric charges  $e, e'$  (in electrostatic units), at a distance  $r$  apart, repel one another with equal and opposite forces of magnitude

$$(6.502) \quad \frac{ee'}{r^2}.$$

If  $e$  and  $e'$  have opposite signs, this force is a force of attraction.

Here we have two examples of *the law of the inverse square*. The law (6.501) governs astronomical phenomena—in particular, the motion of a planet round the sun. The law (6.502) governs atomic phenomena—in particular, the motion of an electron in an atom about the central nucleus. In this case, of course, the charges  $e$ ,  $e'$  have opposite signs, so that the force is one of attraction, as in the gravitational case. It is remarkable that the same form for the law of attraction should hold on such different scales.

The expressions (6.501) and (6.502), combined with Newton's law of motion, constitute two hypotheses regarding phenomena in gravitational and electrostatic fields. For a long time, they were accepted as completely valid from a physical point of view, but that is no longer the case. The modern astronomer knows that gravitational attraction should be discussed in terms of the general theory of relativity, and the physicist insists that problems on the atomic scale belong to quantum mechanics. It would, however, create a completely false impression if we were to say that the law of the inverse square has disappeared from modern science. Nearly all the calculations of astronomers are still based on (6.501) and give results in excellent agreement with observation. Moreover, the physicist often falls back on the simple atomic picture based on (6.502) and Newton's law of motion.

In what follows, we shall discuss the motion of a planet attracted by the sun. Obviously, by a mere change of constant, the same reasoning will apply to the motion of an electron in an atom.

### **Determination of the orbit.**

The sun and a planet are regarded as particles, of masses  $M$  and  $m$ , respectively. The attraction of the sun on the planet, given by (6.501), produces an acceleration  $GM/r^2$ ; and the attraction of the planet on the sun produces an acceleration  $Gm/r^2$ . These accelerations are in the ratio  $M/m$ , which is actually a very large number. Hence, without any serious departure from reality, we may neglect the acceleration of the sun and treat it as if it were at rest. Later we shall see how to treat the problem exactly.

We consider then the case of a particle attracted toward a fixed center by a force  $P$  per unit mass, where

$$(6.503) \quad P = \frac{\mu}{r^2},$$

$\mu$  being some positive constant. The differential equation (6.414) for the orbit now reads

$$(6.504) \quad \frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2}.$$

The general solution is

$$(6.505) \quad u = \frac{\mu}{h^2} + C \cos (\theta - \theta_0),$$

where  $C$  and  $\theta_0$  are constants of integration. *This is, in polar coordinates, the equation of the most general orbit described under a central force varying as the inverse square of the distance.*

The potential energy per unit mass is

$$(6.506) \quad V = \int P dr = -\frac{\mu}{r} = -\mu u,$$

the constant of integration being chosen to make  $V$  vanish at infinity.

Let us now substitute from (6.505) in (6.418), the equation of energy, in order to express the constant  $C$  in terms of  $E$  and  $h$  (the total energy and angular momentum per unit mass). We get

$$C^2 + \frac{\mu^2}{h^4} + 2C \frac{\mu}{h^2} \cos (\theta - \theta_0) = \frac{2}{h^2} \left[ E + \frac{\mu^2}{h^2} + \mu C \cos (\theta - \theta_0) \right],$$

so that

$$(6.507) \quad C^2 = \frac{\mu^2}{h^4} + \frac{2E}{h^2}.$$

By rotating the base line  $\theta = 0$ , we can make  $\theta_0 = 0$  and  $C > 0$  in (6.505); this we shall suppose done. Then the equation (6.505) for the orbit reads

$$(6.508) \quad u = \frac{\mu}{h^2} \left( 1 + \sqrt{1 + \frac{2Eh^2}{\mu^2}} \cos \theta \right).$$

From the focus-directrix property of a conic, we know that its equation in polar coordinates may be written

$$(6.509) \quad u = \frac{1}{l} (1 + e \cos \theta),$$

where  $l$  is the semi-latus-rectum (i.e., half the focal chord parallel to the directrix) and  $e$  the eccentricity;  $\theta$  is measured from the perpendicular dropped from the focus on the directrix. The conic may be of any of the following types:

ellipse ( $e < 1$ ),

parabola ( $e = 1$ ),  
hyperbola ( $e > 1$ ).

In the case of the hyperbola, (6.509) gives only the branch adjacent to the focus.

Comparing (6.508) and (6.509), we note that it is always possible to bring the equations into complete agreement by choosing for  $l$  and  $e$  the values

$$(6.510) \quad l = \frac{h^2}{\mu}, \quad e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}.$$

Accordingly, we may say: *The orbit described by a particle, attracted to a fixed center by a force varying as the inverse square of the distance, is a conic having the center of force for focus. The semi-latus-rectum and the eccentricity are given by (6.510) in terms of the angular momentum and energy per unit mass. The orbit may be of the following types:*

ellipse ( $E < 0$ ),  
parabola ( $E = 0$ ),  
hyperbola ( $E > 0$ ).

The fact that orbits may be classified thus in terms of the total energy is remarkable.

The most important orbits in astronomy (those of the planets) are ellipses. Recurring comets describe orbits which are elongated ellipses, approximating to parabolas. A body with a parabolic or hyperbolic orbit would pass out from the solar system, never to return.

## Constants of the elliptical orbit.

Let us now confine our attention to the elliptical orbit. Since the orbits of the planets are of this type, a great wealth of technical detail has been developed about the elliptical orbit. We shall here give only a brief treatment.

It is evident from (6.509) that the shape and size of an orbit (but not its orientation in space) are determined by the two constants  $l$ ,  $e$ . These are related to the constants  $E$ ,  $h$  by (6.510). Thus, of the various constants which appear in our equations, we are to regard  $\mu$  (the intensity of the force center) as given once for all, whereas the constants  $l$ ,  $e$ ,  $E$ ,  $h$  take different values for different orbits. On account of (6.510), only two of these constants are independent. We may use as an independent pair any two which prove convenient.

Instead of using  $(l, e)$  as fundamental constants, it is better to use  $(a, e)$ , where  $a$  is the semiaxis major of the orbit. Now,

$$(6.511) \quad l = \frac{b^2}{a} = a(1 - e^2),$$

$b$  being the semiaxis minor. We shall refer to  $(a, e)$  as the *geometrical* constants of an orbit and  $(E, h)$  as its *dynamical* constants. The formulas of transformation from one set to the other are as follows:

$$(6.512) \quad \begin{cases} a = -\frac{\mu}{2E}, & e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}; \\ E = -\frac{\mu}{2a}, & h = \sqrt{\mu a(1 - e^2)}. \end{cases}$$

There is a simple formula giving the speed  $q$  at any point of the orbit in terms of the radius vector. By the equation of energy, we have

$$\frac{1}{2}q^2 - \frac{\mu}{r} = E.$$

Substituting for  $E$  from (6.512), we obtain

$$(6.513) \quad q^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right).$$

### The periodic time.

We now ask: How long does the particle take to describe the elliptical orbit? This time is called the *periodic time* ( $\tau$ ). We seek an expression for  $\tau$  in terms of the fundamental constants.

The periodic time cannot be obtained from (6.414), because the time has been eliminated from this equation. We refer instead to (6.408), which gives for the areal velocity

$$\dot{A} = \frac{1}{2}h.$$

If  $F$  is the focus at which the center of force is situated, it follows at once that the particle describes an arc  $VP$ , starting from the vertex<sup>\*</sup>  $V$  nearer to  $F$ , in a time  $2A/h$ , where  $A$  is the area of the sector subtended at  $F$  by this arc (Fig. 80). Following the point  $P$  right round the orbit, we get for the periodic time

$$(6.514) \quad \tau = \frac{2A}{h},$$

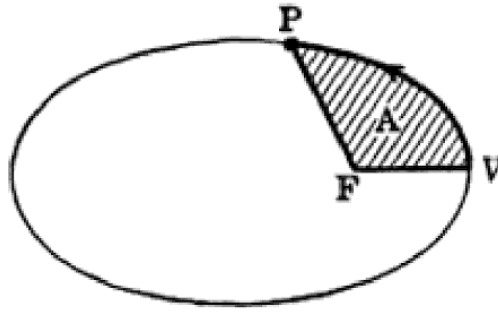


FIG. 80.—The rate of increase of  $A$  is constant.

where  $A$  is now the total area of the ellipse. We might substitute  $A = \pi ab$ ; but, to be systematic, we should express  $\tau$  in terms of either the geometrical constants or the dynamical constants. Since

$$b = a \sqrt{1 - e^2},$$

we obtain, by some easy calculations,

$$(6.515) \quad \tau = 2\pi \sqrt{\frac{a^3}{\mu}} = \frac{2\pi\mu}{\sqrt{(-2E)^3}}.$$

(We remember that  $E < 0$  for the elliptical orbit.)

It is remarkable that the formula involves only one geometrical constant or one dynamical constant. All orbits with the same semiaxis major have the same periodic time; so also have all orbits with the same total energy.

### Kepler's laws.

Before the mathematical theory given above had been developed by Newton, Kepler deduced the following laws of planetary motion from a careful study of the results of astronomical observations:

- I. Each planet describes an ellipse with the sun in one focus.
- II. The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.
- III. The squares of the periodic times of the planets are proportional to the cubes of the semiaxes major of their orbits.

Starting from Newton's law of gravitation, we have shown that all these statements are true. But it is interesting to adopt the historical point of view and face the problem as it presented itself to Newton: *Given Kepler's laws as a statement of fact, what is the law of gravitational attraction?*

Law II tells us that  $h$ —the angular momentum per unit mass—is constant, and hence that the force must be directed toward the sun. From Law I, we



know that the equation of an orbit may be written

$$u = \frac{1}{l} (1 + e \cos \theta).$$

Then by (6.414) the force per unit mass is

$$(6.516) \quad P = h^2 u^2 \left( \frac{d^2 u}{d\theta^2} + u \right) = \frac{h^2 u^2}{l}.$$

Thus for each planet the force varies inversely as the square of the distance. But it remains to prove that the force is of the form

$$(6.517) \quad mP = \frac{m\mu}{r^2},$$

where  $m$  is the mass of the planet and  $\mu$  a constant, the same for all the planets. To show this, we appeal to Law III. We know that for an elliptical orbit, described under a central force directed to a focus,

$$\tau = \frac{2\pi ab}{h},$$

and so

$$\frac{\tau^2}{a^3} = \frac{4\pi^2 b^2}{h^2 a} = \frac{4\pi^2 l}{h^2}.$$

But, by Law III, this is a constant, the same for all planets. Thus  $h^2/l$  is the same for all planets; and so, by (6.516),  $P = \mu u^2$ , where  $\mu$  is the same for all planets. Hence (6.517) is true, and Newton's law of gravitation is thus deduced as a consequence of Kepler's laws.

If Kepler's laws were accurately true, we should have to regard the sun as fixed and the planets as attracted *only* by the sun. More precise measurements show that Kepler's laws are only an approximation and that the inverse-square law of attraction holds for every pair of bodies. It is fortunate that the observations of Kepler's time were crude, because otherwise the simplicity of the law of gravitation would have been obscured.

### **The two-body problem.**

An accurate dynamical treatment of the solar system involves complexities far greater than those encountered in the above discussion. First, the sun is accelerated by the attractions of the planets; secondly, the mutual attractions of the planets influence their motions. A full treatment of the problem belongs to

## IV. Planetary orbits.

(a) The orbit under an attraction  $m\mu/r^2$  is a conic section with one focus at the center of force: ellipse for  $E < 0$ , parabola for  $E = 0$ , hyperbola for  $E > 0$ .

(b) For elliptical orbit

$$(6.612) \quad \frac{l}{r} = 1 + e \cos \theta, \quad l = a(1 - e^2); \quad q^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right);$$

$$(6.613) \quad \begin{cases} E = -\frac{\mu}{2a}, & h = \sqrt{\mu a(1 - e^2)}, \\ \text{periodic time} = \tau = 2\pi \sqrt{\frac{a^3}{\mu}}. \end{cases}$$

## V. Two-body problem.

Relative motion is the same as if one body were held fixed and its mass increased to the sum of the two masses.

### EXERCISES VI

1. A particle is projected upward in a direction inclined at  $60^\circ$  to the horizontal. Show that its velocity when at its greatest height is half its initial velocity. (Neglect the resistance of the air.)

2. A particle of mass  $m$  moves on a straight line under the influence of a force directed toward the origin  $O$  on the line and proportional to the distance from  $O$ ; the force at unit distance is of magnitude  $mk^2$ . The particle passes  $O$  with a velocity  $u$ . If  $x$  is its coordinate at time  $t$  and  $v$  its velocity at that instant, show that  $v^2 + k^2x^2 = u^2$ .

3. Prove by a general argument, not involving any particular law of resistance, that a body thrown vertically upward in a resisting medium will return to the point of projection with a velocity less than that with which it was projected.

4. A gun is mounted on a hill of height  $h$  above a level plain. Show that, if the resistance of the air is neglected, the greatest horizontal range for given muzzle velocity  $V$  is obtained by firing at an angle of elevation  $\theta$  such that

$$\operatorname{cosec}^2 \theta = 2(1 + gh/V^2).$$

5. Find the greatest distance that a stone can be thrown inside a horizontal tunnel 10 feet high with a velocity of projection of 80 feet per second. Find also the corresponding time of flight.

6. In a resisting medium two identical bodies are let fall from the same position at instants separated by an interval  $t$ . Show that the distance between them tends to the limit  $vt$ , where  $v$  is the limiting velocity.

7. Calculate the rate of loss of energy (kinetic + potential) for a damped harmonic oscillator vibrating as in (6.330).

$$(7.104) \quad I = \int r^2 dm,$$

where the integration sign indicates the limit of a process in which the system is divided into a great number of very small parts, and the sum taken;  $dm$  is the mass of an infinitesimal element, and  $r$  is its distance from the line about which the moment of inertia is to be found.

Moment of inertia has the dimensions  $[ML^2]$  and is measured in gm. cm.<sup>2</sup> in the e.g.s. system and in lb. ft.<sup>2</sup> in the f.p.s. system. The square of the radius of gyration has the dimensions [moment of inertia]/[mass], or  $[ML^2]/[M]$ , i.e.,  $[L^2]$ . Thus radius of gyration has dimensions  $[L]$  and so is a length.

Let us now calculate some simple moments of inertia.

*Hoop.* It is evident that the moment of inertia of a thin hoop of mass  $m$  and radius  $a$  about a line through its center perpendicular to its plane is  $ma^2$ .

*Rod.* Let us calculate the moment of inertia of a uniform rod of mass  $m$  and length  $2a$  about a line through its center perpendicular to its length. Taking the rod for  $x$ -axis, with the origin at the center of the rod, the mass of an element  $dx$  is

$$dm = \frac{m dx}{2a}.$$

Hence, by (7.104), we have

$$(7.105) \quad I = \int_{-a}^a x^2 \frac{m dx}{2a} = \frac{1}{3}ma^2.$$

*Rectangular plate.* Consider now a uniform rectangular plate of mass  $m$  and edges of lengths  $2a$ ,  $2b$ . We wish to calculate the moment of inertia about the line in its plane passing through the center and parallel to the edge  $2b$ . We imagine the plate split into thin strips parallel to the edge  $2a$ . Let  $dm$  be the mass of a strip. Then the moment of inertia of the strip is  $\frac{1}{3}a^2 dm$ , by (7.105), and hence the moment of inertia of the whole plate is  $\frac{1}{3}ma^2$ .

*Circular disk.* We wish to find the moment of inertia of a uniform circular disk of mass  $m$  and radius  $a$  about a line through its center perpendicular to its plane. We imagine the disk split up into thin rings by a great number of circles concentric with the boundary. If  $r$ ,  $r + dr$  are the inner and outer radii of a ring, the area of the ring is  $2\pi r dr$ , and its mass is

$$dm = m \cdot \frac{2\pi r dr}{\pi a^2}.$$

The moment of inertia is

$$(7.106) \quad I = \int r^2 dm = \int_0^a \frac{2m}{a^2} r^3 dr = \frac{1}{2}ma^2.$$

*Circular cylinder.* The moment of inertia of a solid circular cylinder about its axis follows immediately from (7.106). For we may imagine the cylinder split up by planes perpendicular to its axis into a great number of thin disks. When we add together their moments of inertia, we get  $I = \frac{1}{2}ma^2$ , where  $m$  is the mass of the cylinder and  $a$  its radius. The length of the cylinder does not appear explicitly in the formula.

*Sphere.* To find the moment of inertia of a solid sphere of mass  $m$  and radius  $a$  about a diameter, we imagine it split into thin circular disks by planes perpendicular to the diameter in question. Figure 82 shows the section of the sphere by a plane through the diameter ( $Ox$ ) about which the moment is to be calculated. If  $\rho$  is the density of the material, the mass of the disk between planes at distances  $x, x + dx$  from the center is

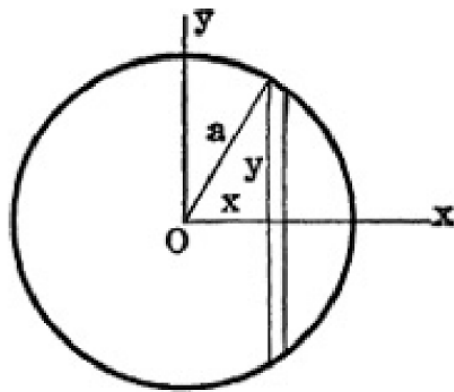


FIG. 82.—Solid sphere split into thin circular disks for the calculation of moment of inertia.

$$\rho \cdot \pi y^2 dx,$$

where  $y$  is the radius of the disk. By (7.106), the moment of inertia of the disk is

$$dI = \frac{1}{2}\pi\rho y^4 dx.$$

But  $y^2 = a^2 - x^2$ , and so

$$I = \frac{1}{2}\pi\rho \int_{-a}^a (a^2 - x^2)^2 dx = \frac{8}{15}\pi\rho a^5.$$

But

$$m = \frac{4}{3}\pi\rho a^3,$$

and so the moment of inertia of the sphere is

$$(7.107) \quad I = \frac{2}{5}ma^2.$$

*Exercise.* Show by the theory of dimensions, without calculations, that the above moments of inertia of the hoop, the rod, the circular disk, the circular cylinder, and the sphere are all necessarily of the form  $Cma^2$ , where  $C$  is a pure number.

### Theorem of parallel axes.

The theorem of parallel axes gives us an easy method of calculating the moment of inertia of a system about any line, when the moment of inertia about a parallel line through the mass center is known. Figure 83 shows a projection onto a plane perpendicular to the two lines,  $O'$  being the projection of the line through the mass center and  $O$  the projection of the other line. Introducing parallel coordinate axes as shown, let  $(a, b)$  be the coordinates of  $O'$  relative to  $O$ . If  $x, y$  are the coordinates of any point relative to the axes  $Oxy$ , and  $x', y'$  the coordinates of the same point relative to  $O'x'y'$ , then

$$(7.108) \quad \begin{cases} x = x' + a, \\ y = y' + b. \end{cases}$$

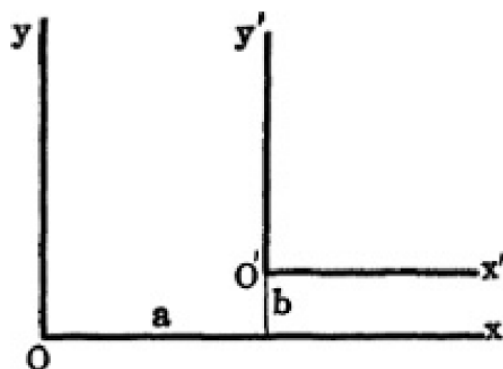


FIG. 83.—Coordinates for the proof of the theorem of parallel axes.

With the notation used at the beginning of this section, the moments of inertia about the lines through  $O$  and  $O'$  are, respectively,

$$(7.109) \quad I = \sum_{i=1}^n m_i(x_i^2 + y_i^2), \quad I' = \sum_{i=1}^n m_i(x_i'^2 + y_i'^2).$$

Then, by (7.108),

$$(7.110) \quad I = \sum_{i=1}^n m_i [(x'_i + a)^2 + (y'_i + b)^2] \\ = I' + m(a^2 + b^2),$$

where  $m$  is the total mass of the system, since

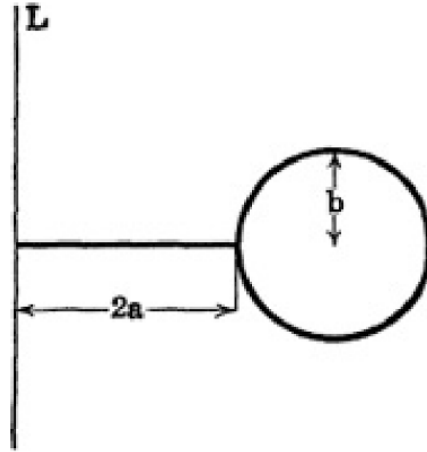


FIG. 84.—Rod and sphere: the moment of inertia about  $L$  is required.

$$\sum_{i=1}^n m_i x'_i = \sum_{i=1}^n m_i y'_i = 0$$

from the definition of mass center. We may state (7.110) in words as follows: *The moment of inertia of a system about an axis  $L$  is equal to the moment of inertia of the same system about an axis through the mass center parallel to  $L$ , together with the moment of inertia about  $L$  of a particle with a mass equal to the total mass of the system, placed at its mass center.*

As an application of this theorem of parallel axes, we note that by (7.105) the moment of inertia of a rod of length  $2a$  about a line through one end, perpendicular to the rod, is  $\frac{4}{3}ma^2$ .

As another application, suppose we wish to find the moment of inertia about the line  $L$  of the apparatus shown in Fig. 84, consisting of a rod of mass  $m$  and length  $2a$ , with a sphere of mass  $M$  and radius  $b$  attached to the end of the rod. From (7.105) and (7.107), combined with the theorem of parallel axes, we obtain at once

$$(7.111) \quad I = \frac{4}{3}ma^2 + M[\frac{2}{5}b^2 + (2a + b)^2].$$

### Theorem of perpendicular axes.

The theorem of perpendicular axes is useful for the calculation of moments



of inertia of plane distributions of matter. Let  $Oxyz$  be rectangular axes, and let there be a distribution of matter in the plane  $z = 0$ . Denoting by  $A$ ,  $B$ ,  $C$  the moments of inertia about the three axes, we have

$$(7.112) \quad A = \sum_{i=1}^n m_i y_i^2, \quad B = \sum_{i=1}^n m_i x_i^2, \quad C = \sum_{i=1}^n m_i (x_i^2 + y_i^2).$$

Obviously,

$$(7.113) \quad C = A + B;$$

this result constitutes the theorem of perpendicular axes.

As an application, suppose we wish to find the moment of inertia of a rectangular plate of edges  $2a$ ,  $2b$ , about a line through its center perpendicular to its plane. Taking the origin at the center, the axes  $Oxy$  parallel to the edges, and the axis  $Oz$  perpendicular to the plate, we have, as already established,

$$(7.114) \quad A = \frac{1}{3}mb^2, \quad B = \frac{1}{3}ma^2.$$

Hence the required moment of inertia is

$$(7.115) \quad C = A + B = \frac{1}{3}m(a^2 + b^2).$$

More information about moments of inertia will be found in Sec. 11.3.

*Exercise.* The moment of inertia of a hoop of mass  $m$  and radius  $a$  about a diameter is  $\frac{1}{2}ma^2$  and that of a circular disk of the same mass and radius about a diameter is  $\frac{1}{4}ma^2$ . Verify these statements.

## Kinetic energy and angular momentum.

As we have seen in Sec. 5.2, kinetic energy and angular momentum play an important part in the dynamics of systems. We shall now show how these quantities are to be calculated when the system is a rigid body moving parallel to a plane.

Let us first suppose that the rigid body is *rotating about a fixed axis*. Let  $\omega$  be the instantaneous value of the angular velocity. Then the kinetic energy of a particle of mass  $m$ , situated at a distance  $r_i$  from the axis is  $\frac{1}{2}m_i r_i^2 \omega^2$ , and so the kinetic energy of the rigid body (supposed to consist of  $n$  particles) is

$$(7.116) \quad T = \frac{1}{2}\omega^2 \sum_{i=1}^n m_i r_i^2 = \frac{1}{2}I\omega^2,$$

where  $I$  is the moment of inertia about the axis. The angular momentum of a particle about the axis of rotation is  $m_i r_i^2 \omega$ , and so the angular momentum of the rigid body is

$$(7.117) \quad h = \omega \sum_{i=1}^n m_i r_i^2 = I\omega.$$

Let us now suppose that the rigid body no longer rotates about a fixed axis but moves in a general manner parallel to a fixed fundamental plane (cf. Sec. 4.2). Let us imagine an observer traveling with one of the particles ( $A$ ) of the rigid body and observing the motion of the particles relative to him. He can compute a *relative* kinetic energy and a *relative* angular momentum about a line through  $A$  perpendicular to the fundamental plane, using in these computations the velocities of the particles relative to him. Since relative to him the body rotates with angular velocity  $\omega$  about a fixed axis through  $A$ , the formal calculations are precisely as above and lead to formulas (7.116) and (7.117) for the relative kinetic energy and angular momentum.

Although this is true for any particle  $A$  of the body, the results are most useful when  $A$  is the mass center. Let us restate them: *The kinetic energy and angular momentum, both relative to the mass center, of a rigid body moving parallel to a plane are*

$$(7.118) \quad T = \frac{1}{2}I\omega^2, \quad h = I\omega,$$

where  $\omega$  is the angular velocity of the body and  $I$  its moment of inertia about an axis through the mass center perpendicular to the plane of motion.

The following theorem of König enables us to complete the calculation of the kinetic energy of a rigid body moving parallel to a plane. We shall prove it in general three-dimensional form.

**THEOREM OF KÖNIG.** *The kinetic energy of a moving system is equal to the sum of (i) the kinetic energy of a fictitious particle moving with the mass center and having a mass equal to the total mass of the system and (ii) the kinetic energy of the motion relative to the mass center.*

Let  $Oxyz$  be fixed axes and  $O'x'y'z'$  parallel axes through the mass center. Let  $\bar{x}, \bar{y}, \bar{z}$  be the coordinates of the mass center referred to  $Oxyz$ . Then, for any particle, we have

$$(7.119) \quad x_i = \bar{x} + x'_i, \quad y_i = \bar{y} + y'_i, \quad z_i = \bar{z} + z'_i.$$

The kinetic energy of the system is

$$(7.120) \quad T = \frac{1}{2} \sum_{i=1}^n m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2).$$

Let us differentiate (7.119) and substitute for  $\dot{x}_i$ ,  $\dot{y}_i$ ,  $\dot{z}_i$  in (7.120). Certain terms vanish on account of the relations

$$(7.121) \quad \sum_{i=1}^n m_i \dot{x}'_i = \sum_{i=1}^n m_i \dot{y}'_i = \sum_{i=1}^n m_i \dot{z}'_i = 0,$$

which are consequences of the definition of mass center given in Sec. 3.1. We obtain

$$(7.122) \quad T = \frac{1}{2} m (\dot{\hat{x}}^2 + \dot{\hat{y}}^2 + \dot{\hat{z}}^2) + \frac{1}{2} \sum_{i=1}^n m_i (\dot{x}'_i{}^2 + \dot{y}'_i{}^2 + \dot{z}'_i{}^2),$$

where  $m = \sum_{i=1}^n m_i$ , the total mass of the system. Thus the theorem is proved.

It is convenient to refer to the kinetic energy of the fictitious particle as the “kinetic energy of the mass center,” so that our result reads

$$(7.123) \quad T = T_0 + T',$$

where  $T_0$  is the kinetic energy of the mass center and  $T'$  the kinetic energy relative to the mass center. This general result holds even though the system is not a rigid body.

In the case of a rigid body, we have the important formula

$$(7.124) \quad T = \frac{1}{2} m q^2 + \frac{1}{2} I \omega^2,$$

where  $m$  = mass of body,

$q$  = speed of mass center,

$I$  = moment of inertia about mass center,\*

$\omega$  = angular velocity.

*Exercise.* Find the kinetic energy of a disk of mass  $m$  and radius  $a$ , rolling along the ground with speed  $q$ .

## 7.2. RIGID BODY ROTATING ABOUT A FIXED AXIS

## General methods.

In Sec. 5.2 we developed the principle of angular momentum (5.214) and the principle of energy (5.223). Let us insert in these equations the values of  $h$  and  $T$  given in (7.117) and (7.116); then we have

$$(7.201) \quad I\dot{\omega} = N \quad (\text{principle of angular momentum}),$$

$$(7.202) \quad \frac{1}{2}I\omega^2 + V = E \quad (\text{principle of energy}).$$

These equations represent the two fundamental methods of finding the motion of a rigid body which turns about a fixed axis. Let us recall the meanings of the terms:

$I$  = moment of inertia about the fixed axis,

$\omega$  = angular velocity,

$N$  = moment of external forces about the fixed axis, or *torque*.

$V$  = potential energy,

$E$  = total energy (a constant).

It must be remembered that (7.201) is always valid; (7.202), on the other hand, holds only when the system is conservative. It would not hold, for example, if there were a frictional torque.

## Flywheels.

Let us consider a flywheel rotating about a fixed axis which passes through its mass center. Gravity contributes nothing to the moment about the axis and so does not influence the motion. We suppose the torque  $N$  supplied by a motor or brakes. Since the forces involved here will not, in general, be conservative, we use (7.201) as the equation of motion; so we write

$$(7.203) \quad I\dot{\omega} = N.$$

We may note the resemblance between this equation and the equation of motion of a particle moving on a straight line,

$$m\dot{u} = X;$$

mass corresponds to moment of inertia, linear velocity to angular velocity, force to torque. This mathematical similarity may be used to solve a problem in the dynamics of a rotating flywheel, when the solution of the analogous problem for a particle moving on a straight line is already known.

If the torque  $N$  is constant, (7.203) gives

$$(7.204) \quad \omega = \frac{N}{I} t + A,$$

where  $A$  is a constant of integration; hence, if  $\theta$  is the angle turned through, we have  $\dot{\theta} = \omega$  to and

$$(7.205) \quad \theta = \frac{N}{2I} t^2 + At + B,$$

where  $B$  is another constant of integration. This motion is analogous to the motion of a particle under a constant force.

The importance of the flywheel in machinery lies in its capacity to smooth out motion. In a steam or gasoline engine the torque is not uniform, and without a flywheel (or something equivalent) the motion would be jerky. As an illustration, let us work out the case where a flywheel of moment of inertia  $I$  is under the action of a torque with a fluctuating part,

$$N = N_0 + N_1 \cos ct, \quad (N_0, N_1 \text{ constants}),$$

and a load proportional to angular velocity. The equation of motion is

$$(7.206) \quad I\dot{\omega} = N_0 + N_1 \cos ct - L\omega,$$

the last term corresponding to the load; thus

$$(7.207) \quad \dot{\omega} + \frac{L}{I} \omega = \frac{1}{I} (N_0 + N_1 \cos ct).$$

This is a standard type of differential equation, which has the integrating factor  $e^{Lt/I}$ ; the solution is

$$(7.208) \quad \omega = e^{-Lt/I} \left[ A + \frac{1}{I} \int e^{Lt/I} (N_0 + N_1 \cos ct) dt \right],$$

where  $A$  is a constant of integration. Using  $R$  to denote “real part of,” we have

$$(7.209) \quad \begin{aligned} \int e^{Lt/I} \cos ct \, dt &= R \int e^{(L/I + ic)t} \, dt \\ &= R \frac{1}{(L/I) + ic} e^{(L/I + ic)t} \\ &= e^{Lt/I} R \frac{(L/I) - ic}{(L/I)^2 + c^2} (\cos ct + i \sin ct) \\ &= e^{Lt/I} \frac{\cos(ct + \epsilon)}{[(L/I)^2 + c^2]^{1/2}}, \end{aligned}$$

where  $\epsilon$  is a constant; its value is of no present interest. Hence,

$$(7.210) \quad \omega = Ae^{-Lt/I} + \frac{N_0}{L} + \frac{N_1}{I[(L/I)^2 + c^2]^{\frac{1}{2}}} \cos(ct + \epsilon).$$

The first term dies away as  $t$  increases. The ratio of the amplitude of the third term to the second term is

$$(7.211) \quad \frac{N_1}{N_0} \cdot \frac{L}{I[(L/I)^2 + c^2]^{\frac{1}{2}}}.$$

By increasing the moment of inertia of the flywheel, we can make this ratio as small as we please and so approximate to the steady motion  $\omega = N_0/L$ , even though the fluctuating part  $N_1 \cos ct$  may be greater in magnitude than the steady torque  $N_0$ .

### The compound pendulum.

In Sec. 6.3, we discussed the small oscillations of a simple pendulum, consisting of a heavy particle attached to a fixed point by a light string. We shall now discuss the *compound pendulum*, which is a rigid body free to oscillate under the influence of gravity about a fixed horizontal axis. We shall use the equation of energy (7.202), but the results may be obtained with equal ease from (7.201).

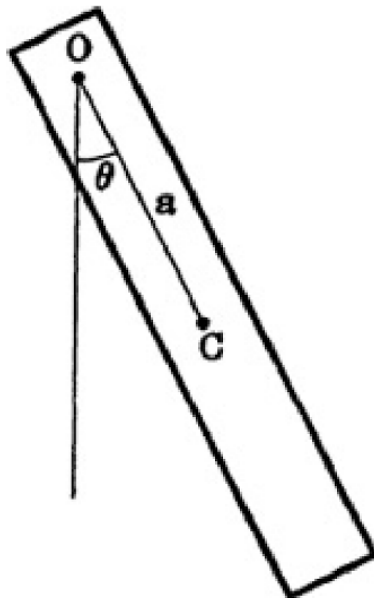


FIG. 85.—A compound pendulum.

In Fig. 85 the plane of the paper is the plane through the mass center  $C$  perpendicular to the axis of suspension. The axis cuts it at  $O$ , which is called the *point of suspension*. We shall use the following notation:

$$a = OC,$$



$m$  = mass of pendulum,

$k_c$  = radius of gyration about  $C$ ,

$k_0$  = radius of gyration about  $O$ .

If  $\theta$  denotes the inclination of  $OC$  to the vertical, the potential energy is

$$V = -mga \cos \theta,$$

and (7.202) gives

$$(7.212) \quad \frac{1}{2}mk_0^2\dot{\theta}^2 - mga \cos \theta = E,$$

where  $E$  is a constant.

This equation gives the angular velocity at any position, when  $E$  has been found from the initial conditions.

For example, if the pendulum starts with  $C$  directly below  $O$  and with angular velocity  $\omega_0$ , we have

$$E = \frac{1}{2}mk_0^2\omega_0^2 - mga;$$

equation (7.212) gives

$$(7.213) \quad \dot{\theta}^2 = \omega_0^2 - \frac{4ga}{k_0^2} \sin^2 \frac{1}{2}\theta.$$

This will vanish when  $\theta$  takes the values  $\pm \alpha$ , where

$$\sin^2 \frac{1}{2}\alpha = \frac{k_0^2\omega_0^2}{4ga},$$

and so the pendulum oscillates through the range  $(-\alpha, \alpha)$ .

Since  $\sin^2 \frac{1}{2}\theta$  cannot exceed unity, it is evident from (7.213) that  $\dot{\theta}$  never vanishes if

$$\omega_0^2 > \frac{4ga}{k_0^2};$$

if started with such an angular velocity, the pendulum travels right around.

Differentiation of (7.212) gives

$$(7.214) \quad k_0^2\ddot{\theta} + ga \sin \theta = 0,$$

as an alternative form for the equation of motion of a compound pendulum. Had we used (7.201), we should have obtained this equation directly without

If we change the position of the point of suspension in the body,  $a$  changes but  $k_c$  does not change. We see that  $\tau$  tends to infinity if  $a$  tends to zero or if  $a$  tends to infinity. Thus, we can obtain very slow oscillations by moving the point of suspension close to the mass center or far from it.

On differentiating (7.220) with respect to  $a$ , we get

$$\frac{d}{da}(\tau^2) = \frac{4\pi^2}{g} \left( 1 - \frac{k_c^2}{a^2} \right).$$

Thus *the periodic time is a minimum when the point of suspension is at a distance from the mass center equal to the radius of gyration about the mass center.*

We now ask whether it is possible to shift the point of suspension from a position  $O$  to a new position  $O'$  (Fig. 86) on the line  $OC$  without changing the periodic time. With  $OC = a$ ,  $CO' = b$ , the condition for equality of periodic times is, by (7.220),

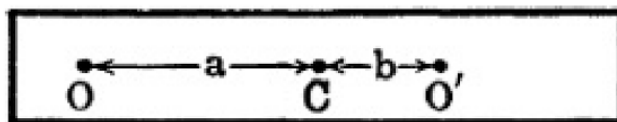


FIG. 86.—The periodic time is the same whether  $O$  or  $O'$  is used as point of suspension.

$$a + \frac{k_c^2}{a} = b + \frac{k_c^2}{b},$$

which is satisfied if

$$(7.221) \quad ab = k_c^2.$$

The point  $O'$ , related in this way to the point of suspension  $O$ , is called the *center of oscillation*.

### 7.3. GENERAL MOTION OF A RIGID BODY PARALLEL TO A FIXED PLANE

#### General methods.

Probably the most useful principle available for the solution of problems in mechanics is the principle of energy in the form (5.223), namely

$$(7.301) \quad T + V = E.$$

One must of course make sure, before attempting to apply this principle, that

initial position at rest at  $t = 0$ , and  $\theta$  the angle through which it has turned. Then, by the condition of rolling,

$$(7.304) \quad x = a\theta.$$

If  $k$  is the radius of gyration of the cylinder about its axis, its kinetic energy is

$$(7.305) \quad T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mk^2\dot{\theta}^2,$$

or, by (7.304),

$$(7.306) \quad T = \frac{1}{2}m \left( 1 + \frac{k^2}{a^2} \right) \dot{x}^2.$$

The potential energy is

$$(7.307) \quad V = -mgx \sin \alpha.$$

Hence, by (7.301),

$$(7.308) \quad \frac{1}{2}m \left( 1 + \frac{k^2}{a^2} \right) \dot{x}^2 - mgx \sin \alpha = E,$$

where  $E$  is the constant total energy. Actually  $E = 0$ , since  $x = \dot{x} = 0$  for  $t = 0$ . Differentiating (7.308) with respect to  $t$ , we obtain

$$(7.309) \quad x = \frac{g \sin \alpha}{1 + (k^2/a^2)}.$$

Thus the cylinder rolls down the inclined plane with a constant acceleration.

A particle would slide down a smooth plane of inclination  $\alpha$  with an acceleration  $g \sin \alpha$ . The value given by (7.309) is always less than  $g \sin \alpha$ , except in the limiting case  $k = 0$ , which corresponds to a concentration of all the mass of the cylinder on its axis. If the cylinder is a thin shell, we have  $k = a$ , and hence

$$(7.310) \quad \ddot{x} = \frac{1}{2}g \sin \alpha.$$

If the cylinder is solid and uniform, we have  $k^2 = \frac{1}{2}a^2$ , and hence

$$(7.311) \quad \ddot{x} = \frac{2}{3}g \sin \alpha.$$

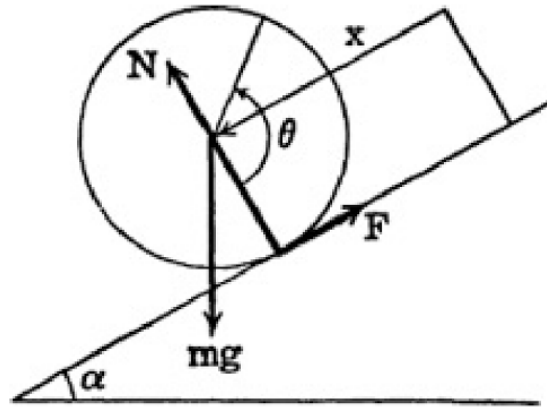


FIG. 88.—External forces acting on cylinder.

The method of energy does not tell us the reaction between the cylinder and the plane, or how rough the plane must be in order that slipping may be avoided. To find out these things, we turn to the principles of linear and angular momentum, using (7.303).

The reaction of the plane on the cylinder may be resolved into a normal component  $N$  and a component  $F$  in the plane (Fig. 88). These forces, with the weight  $mg$ , form the complete system of external forces. Thus, we have

$$(7.312) \quad \begin{cases} m\ddot{x} = mg \sin \alpha - F, \\ 0 = mg \cos \alpha - N, \\ mk^2\ddot{\theta} = Fa, \end{cases}$$

the second equation coming from resolution perpendicular to the plane. Using (7.304) and eliminating  $F$ , we get

$$(7.313) \quad \ddot{x} = \frac{g \sin \alpha}{1 + (k^2/a^2)},$$

as in (7.309). Hence the components of the reaction are

$$(7.314) \quad F = m \frac{k^2}{a^2} \ddot{x} = \frac{mgk^2 \sin \alpha}{a^2 + k^2}, \quad N = mg \cos \alpha.$$

In order that rolling may occur without slipping, we must have  $F/N \leq \mu$ , or

$$(7.315) \quad \mu \geq \frac{k^2 \tan \alpha}{a^2 + k^2},$$

where  $\mu$  is the coefficient of static friction as in (3.202).

### Self-propelled vehicle.

Consider an automobile (Fig. 89). The external forces acting on it are

Figure 90 shows the external forces acting on an automobile, driven through the rear wheels on the right, air resistance being neglected.

Let  $h$  = height of mass center above ground,

$b_1$  = distance of front axle in front of mass center,

$b_2$  = distance of rear axle behind mass center,

$N_1$  = resultant of vertical reactions at two front wheels,

$N_2$  = resultant of vertical reactions at two rear wheels,

$F$  = resultant of frictional forces at two rear wheels.

Then,

$$(7.316) \quad \begin{cases} mf = F, \\ 0 = N_1 + N_2 - mg, \\ 0 = b_2 N_2 - b_1 N_1 - hF. \end{cases}$$

Solving for  $N_1$ ,  $N_2$ ,  $F$ , we obtain

$$(7.317) \quad N_1 = m \frac{gb_2 - fh}{b_1 + b_2}, \quad N_2 = m \frac{gb_1 + fh}{b_1 + b_2}, \quad F = mf.$$

By the law of static friction (3.202)  $F/N_2 \leq \mu$ , and so

$$(7.318) \quad \frac{f(b_1 + b_2)}{gb_1 + fh} \leq \mu,$$

or

$$(7.319) \quad f \leq \frac{\mu gb_1}{b_1 + b_2 - \mu h}.$$

This fraction represents the greatest acceleration possible without slipping.

The maximum negative acceleration obtainable by the application of brakes, without slipping between the tires and the ground, may be found in a similar way.

### Internal reactions.

The principles of linear and angular momentum do not involve the internal reactions between the particles of a rigid body. Nevertheless these reactions exist, and when they become excessive the body may break. As we shall now see, the principles of linear and angular momentum may be used to find the reactions. D'Alembert's principle (cf. Sec. 5.2) may also be used.

principle of angular momentum,

$$(7.323) \quad M - \frac{1}{2}S(l - r) = \frac{mk^2\dot{\omega}}{l} (l - r).$$

Thus,

$$(7.324) \quad M = \frac{m\dot{\omega}(l - r)}{l} [k^2 + \frac{1}{4}(l^2 - r^2)].$$

But

$$k^2 = \frac{1}{12}(l - r)^2,$$

and so

$$(7.325) \quad M = \frac{1}{8} \frac{m\dot{\omega}}{l} (l - r)^2(2l + r).$$

*Exercise.* Find where the rod is most likely to break, assuming that this occurs where the bending moment is greatest.

## 7.4. NORMAL MODES OF VIBRATION

### Degrees of freedom.

The position of a simple pendulum is determined by the value of one variable, namely, its inclination to the vertical, or the horizontal component of the displacement of the bob. A system whose position may be specified by one variable or coordinate is said to be a system with *one degree of freedom*.

A rod which can move in a plane, with one end constrained to move on a fixed line, can be described as to position by two variables, namely, the distance of the constrained end from a fixed point on the constraining line and the inclination of the rod to the line. Each of these variables can take arbitrary values. A system whose position may be specified by two arbitrary and independent variables or coordinates is said to be a system with *two degrees of freedom*.

Similarly, there are systems with  $n$  degrees of freedom, where  $n = 3, 4, \dots$

In Sec. 6.3 we discussed the oscillations or vibrations of a simple pendulum and in Sec. 7.2 those of a compound pendulum. Each of these is a system with one degree of freedom. We now proceed to discuss systems with two degrees of freedom.

### Particles on a stretched string.



$$(7.405) \quad \begin{cases} A(-n^2 + 2k^2) - Bk^2 = 0, \\ -Ak^2 + B(-n^2 + 2k^2) = 0. \end{cases}$$

Elimination of  $A$  and  $B$  gives the determinantal equation

$$(7.406) \quad \begin{vmatrix} n^2 - 2k^2 & k^2 \\ k^2 & n^2 - 2k^2 \end{vmatrix} = 0,$$

or

$$(7.407) \quad n^4 - 4k^2n^2 + 3k^4 = 0;$$

the solutions are  $n_1, n_2$ , where

$$(7.408) \quad n_1^2 = k^2, \quad n_2^2 = 3k^2.$$

When  $n$  is known, either of the equations (7.405) gives, for the ratio  $B/A$ ,

$$(7.409) \quad \frac{B}{A} = 2 - \frac{n^2}{k^2}.$$

Thus for  $n = n_1$ ,  $B/A = 1$ ; and for  $n = n_2$ ,  $B/A = -1$ .

Hence if  $A_1, \epsilon_1$  are arbitrary constants, the following is a solution of (7.403):

$$(7.410) \quad x = A_1 \cos(kt + \epsilon_1), \quad y = A_1 \cos(kt + \epsilon_1).$$

Also, if  $A_2, \epsilon_2$  are arbitrary constants, the following is a solution of (7.403):

$$(7.411) \quad x = A_2 \cos(kt\sqrt{3} + \epsilon_2), \quad y = -A_2 \cos(kt\sqrt{3} + \epsilon_2).$$

Thus, (7.410) and (7.411) represent possible vibrations of the particles. The most general vibration is given by adding these expressions, thus:

$$(7.412) \quad \begin{cases} x = A_1 \cos(kt + \epsilon_1) + A_2 \cos(kt\sqrt{3} + \epsilon_2), \\ y = A_1 \cos(kt + \epsilon_1) - A_2 \cos(kt\sqrt{3} + \epsilon_2). \end{cases}$$

We know that this is the general solution, because it contains four constants of integration which may be chosen to satisfy initial conditions corresponding to given positions and velocities of the particles at  $t = 0$ .

Let us suppose, for example, that when  $t = 0$  we have

$$x = y = \dot{x} = 0, \quad \dot{y} = v.$$

This corresponds to the case where the motion is started by giving a blow to  $D$ .

Putting  $t = 0$  in (7.412) and the equations obtained by differentiating (7.412), we obtain

$$(7.413) \quad \begin{cases} A_1 \cos \epsilon_1 + A_2 \cos \epsilon_2 = 0, \\ A_1 \cos \epsilon_1 - A_2 \cos \epsilon_2 = 0, \\ -A_1 \sin \epsilon_1 - A_2 \sqrt{3} \sin \epsilon_2 = 0, \\ -A_1 \sin \epsilon_1 + A_2 \sqrt{3} \sin \epsilon_2 = \frac{v}{k}, \end{cases}$$

which are four equations for  $A_1, A_2, \epsilon_1, \epsilon_2$ . The solution is

$$(7.414) \quad \epsilon_1 = \epsilon_2 = \frac{1}{2}\pi, \quad A_1 = -\frac{v}{2k}, \quad A_2 = \frac{v}{2k\sqrt{3}},$$

so that the motion of the particles is given by

$$(7.415) \quad \begin{cases} x = \frac{v}{2k} \left[ \sin kt - \frac{1}{\sqrt{3}} \sin (kt \sqrt{3}) \right], \\ y = \frac{v}{2k} \left[ \sin kt + \frac{1}{\sqrt{3}} \sin (kt \sqrt{3}) \right]. \end{cases}$$

This motion is complicated, but the simple harmonic motions of which it is composed are easy to describe. These simple harmonic vibrations are called *normal modes of vibrations*. They are executed by a vibrating system when the initial conditions are properly chosen.

The vibration given in (7.415) is not a normal mode of vibration, nor in general is that given by (7.412). But if the initial conditions are chosen so that  $A_2 = 0$ , we have the normal mode of vibration (7.410), whereas, if the initial conditions are chosen so that  $A_1 = 0$ , we have the normal mode of vibration (7.411). The periodic times and frequencies of normal modes are called *normal periods* and *normal frequencies*. In the above problem the normal periods are

$$\frac{2\pi}{k}, \quad \frac{2\pi}{k\sqrt{3}}.$$

When a system is executing a normal vibration, its configurations are usually simple to describe. Thus, in the mode (7.410) we have  $x = y$ , and in (7.411) we have  $x = -y$ . Typical configurations for the normal modes are shown in [Figs. 93a](#) and [b](#).

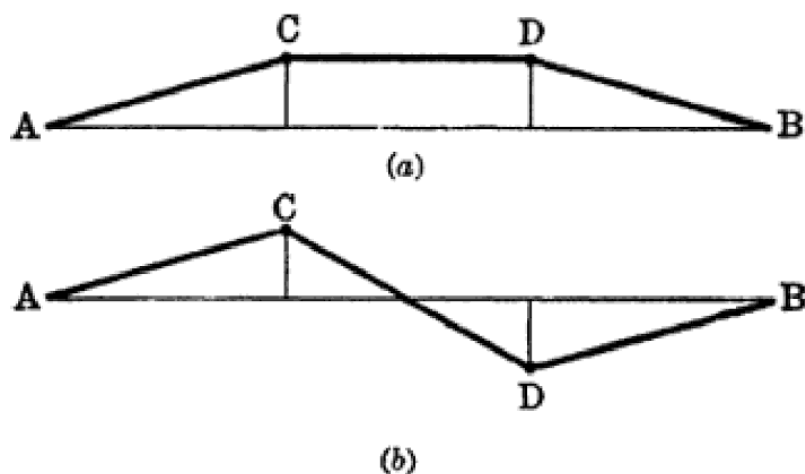


FIG. 93.—(a) Loaded string vibrating in first normal mode. (b) Loaded string vibrating in second normal mode.

We have been discussing the vibrations of a particular system—two particles on a taut string. All problems of vibration have certain features in common; and although we shall not attempt here to prove these facts, it will be useful to sum them up:

(i) A vibration may be regarded as a superposition, or addition, of simple harmonic vibrations.

(ii) Each simple harmonic vibration is called a normal mode of vibration. It is possible to make a system vibrate in a normal mode by starting with suitable initial conditions.

(iii) The periods and frequencies of the normal modes are called the normal periods and frequencies.

(iv) The number of normal modes is equal to the number of degrees of freedom of the system.

(v) The normal periods are found by solving a determinantal equation, e.g., (7.406).

### Vibrations of a particle in a plane.

As another example of a vibrating system with two degrees of freedom, let us consider a particle which moves in a plane in a field of force such that the potential energy per unit mass is

$$(7.416) \quad V = \frac{1}{2}(ax^2 + 2hxy + by^2),$$

where  $a$ ,  $h$ ,  $b$  are constants. The force components per unit mass are then

$$(7.417) \quad X = -(ax + hy), \quad Y = -(hx + by).$$

These vanish at the origin, which is therefore a position of equilibrium.

The equations of motion are

$$(7.418) \quad \ddot{x} = -ax - hy, \quad \ddot{y} = -hx - by.$$

Trying a solution

$$(7.419) \quad x = A \cos (nt + \epsilon), \quad y = B \cos (nt + \epsilon),$$

we see that (7.418) are satisfied provided  $A, B, n$  satisfy

$$(7.420) \quad \begin{cases} A(n^2 - a) - Bh = 0, \\ -Ah + B(n^2 - b) = 0. \end{cases}$$

Hence,  $n$  must satisfy the determinantal equation

$$(7.421) \quad \begin{vmatrix} n^2 - a & -h \\ -h & n^2 - b \end{vmatrix} = 0,$$

or

$$(7.422) \quad n^4 - n^2(a + b) + ab - h^2 = 0;$$

the solutions are  $n_1, n_2$ , where

$$(7.423) \quad \begin{cases} n_1^2 = \frac{1}{2}[(a + b) + \sqrt{(a - b)^2 + 4h^2}], \\ n_2^2 = \frac{1}{2}[(a + b) - \sqrt{(a - b)^2 + 4h^2}]. \end{cases}$$

If one of these values should be negative, the corresponding  $n$  would be imaginary, and the solution (7.419) would contain hyperbolic instead of trigonometrical functions. This case will be discussed in Sec. 7.5; for the present, we assume that  $n_1, n_2$  are real.

Then the normal periods are  $2\pi/n_1, 2\pi/n_2$ , and the normal modes of vibration are

$$(7.424) \quad x = A_1 \cos (n_1 t + \epsilon_1), \quad y = \frac{n_1^2 - a}{h} A_1 \cos (n_1 t + \epsilon_1),$$

and

$$(7.425) \quad x = A_2 \cos (n_2 t + \epsilon_2), \quad y = \frac{n_2^2 - a}{h} A_2 \cos (n_2 t + \epsilon_2),$$

where  $A_1, A_2, \epsilon_1, \epsilon_2$  are arbitrary constants. The general motion is found by adding the solutions (7.424) and (7.425), just as we added (7.410) and (7.411).

The preceding discussion is actually more general than might appear. Let us

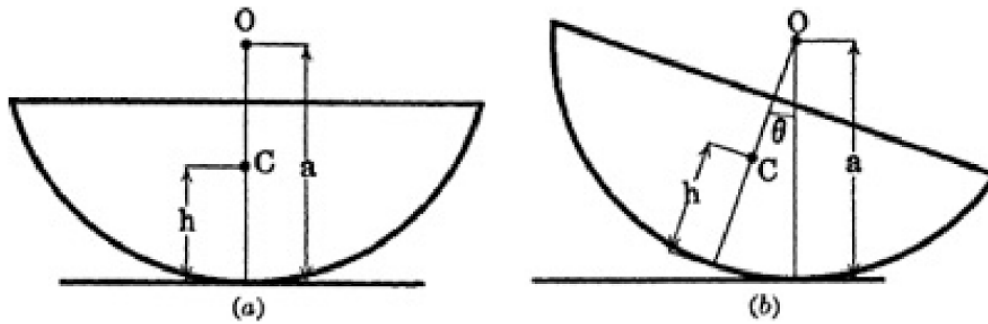


FIG. 95.—Cylindrical body rolling on a horizontal plane: (a) position of equilibrium, (b) displaced position.

If, instead of a needle, we try to balance a body with a rounded base, it is not immediately evident whether the equilibrium in the position of balancing is stable or not. But the condition of minimum energy gives us an easy test. We shall confine our attention to cases where the possible motion of the body is two-dimensional.

Figures 95a and b show end views of a cylindrical body in contact with a rough horizontal plane; the lower part of the section is a circular arc of radius  $a$ . In equilibrium (Fig. 95a) the center of gravity  $C$  must lie vertically above the point of contact, since the body is in equilibrium under two forces (the weight and the reaction) and their lines of action must coincide. Let  $h$  be the height of the center of gravity above the point of contact.

Figure 95b shows a displaced position, in which the body has been turned through an angle  $\theta$ . If  $W$  is the weight of the body, the potential energy is

$$(7.513) \quad V = W[a - (a - h) \cos \theta].$$

Hence,

$$(7.514) \quad \left( \frac{d^2 V}{d\theta^2} \right)_{\theta=0} = W(a - h).$$

Thus the equilibrium is stable if, and only if,  $a > h$ , i.e., if the center of gravity lies below the center of the circle.

Let us now consider a more general problem, which includes the preceding as a special case. Let  $A$  be a cylinder of any section, balanced on a fixed cylinder  $A'$ , the contact being rough and the common tangent horizontal (Fig. 96a).  $C$  is the center of gravity of  $A$ .  $D, D'$  are the centers of curvature of the sections of the cylinders at the point of contact, and  $\rho, \rho'$  are the radii of curvature; the height of  $C$  above the point of contact is  $h$ .

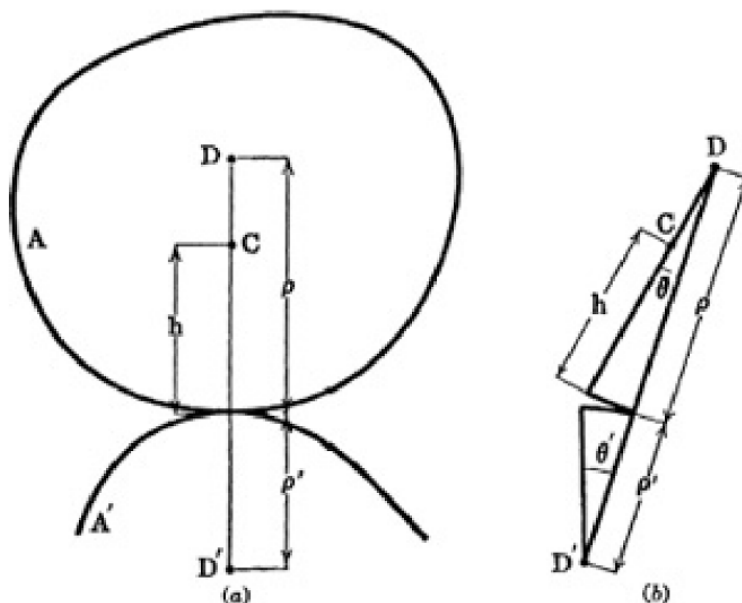


FIG. 96.—One cylinder rolling on another: (a) position of equilibrium, (b) displaced position.

Figure 96b shows a displaced position, in which the point of contact has moved through a small angle  $\theta'$  about  $D'$ , and the line  $DC$  now makes a small angle  $\theta$  with  $DD'$ . Since we are concerned only with small values of  $\theta$  and  $\theta'$ , it is legitimate to regard the sections in the neighborhood of the point of contact as circular arcs of radii  $\rho$  and  $\rho'$ . Then, by the condition of rolling,

$$(7.515) \quad \rho\theta = \rho'\theta'.$$

If  $W$  is the weight of  $A$ , the potential energy is

$$(7.516) \quad V = W[(\rho' + \rho) \cos \theta' - (\rho - h) \cos (\theta + \theta')],$$

or, since  $\theta, \theta'$  are small,

$$(7.517) \quad V = \frac{1}{2}W[(\rho - h)(\theta + \theta')^2 - (\rho' + \rho)\theta'^2] + C,$$

approximately, where  $C$  is a constant. By (7.515), this may be written

$$(7.518) \quad V = \frac{1}{2}W\theta'^2 \left[ (\rho - h) \left( 1 + \frac{\rho'}{\rho} \right)^2 - (\rho' + \rho) \right] + C.$$

The condition that this shall be a minimum for  $\theta' = 0$  is

$$(7.519) \quad (\rho - h) \left( 1 + \frac{\rho'}{\rho} \right)^2 - (\rho' + \rho) > 0,$$

or, equivalently,



$$(7.520) \quad h < \frac{\rho\rho'}{\rho + \rho'}.$$

This is the condition of stability. On letting  $\rho' \rightarrow \infty$ , it reads  $h < \rho$ , agreeing with the result established earlier. If on the other hand we let  $\rho \rightarrow \infty$ , we get  $h < \rho'$  as the condition for the stability of a body with a flat base balanced on a cylinder with radius of curvature  $\rho'$ .

By means of the principle of energy, it is easy to find the period of small oscillations of a stable balanced system when disturbed. For example, the body shown in Fig. 95 has the potential energy given by (7.513). To convert to dynamical units, we put  $W = mg$ , where  $m$  is the mass of the cylinder. Thus, when  $\theta$  is small, the potential energy is approximately

$$(7.521) \quad V = mgh + \frac{1}{2}mg(a - h)\theta^2.$$

Since, at any instant, the cylinder is turning about the line of contact, the velocity of the mass center is approximately  $h\dot{\theta}$ ; the angular velocity of the cylinder is  $\dot{\theta}$ . Hence (7.302) gives

$$(7.522) \quad \frac{1}{2}mh^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 + mgh + \frac{1}{2}mg(a - h)\theta^2 = E,$$

where  $I$  is the moment of inertia about a line through  $C$  parallel to the generators. If we write

$$(7.523) \quad k^2 = h^2 + \frac{I}{m},$$

and differentiate (7.522), we get

$$(7.524) \quad k^2\ddot{\theta} + g(a - h)\theta = 0;$$

this gives a simple harmonic motion with period

$$(7.525) \quad \tau = 2\pi k / \sqrt{g(a - h)}.$$

## 7.6. SUMMARY OF APPLICATIONS IN PLANE DYNAMICS— MOTION OF A RIGID BODY AND OF A SYSTEM

### I. Moments of inertia.

(a) Definitions:

$$(7.601) \quad I = \sum_{i=1}^n m_i r_i^2 \quad \text{or} \quad I = \int r^2 dm = \iiint \rho(x^2 + y^2) dx dy dz;$$

$$mk^2 = I.$$

(b) Devices for calculation:

- (i) Theorem of parallel axes (all moments of inertia follow immediately when those for axes through mass center are known).
- (ii) Theorem of perpendicular axes (for a plane distribution, the moments of inertia about axes perpendicular to the plane follow immediately when those for axes in the plane are known).

(c) Standard results:

Body	Axis	$k^2$
Hoop of radius $a$ .....	Perpendicular to plane at center	$a^2$
Circular disk of radius $a$ .....	Perpendicular to plane at center	$\frac{1}{2}a^2$
Circular cylinder of radius $a$ .....	Geometrical axis	$\frac{1}{2}a^2$
Rod of length $2a$ .....	Perpendicular to rod at center	$\frac{1}{3}a^2$
Rectangular plate of edges $2a, 2b$ ..	Through center parallel to edge $2b$	$\frac{1}{3}a^2$
	Perpendicular to plate at center	$\frac{1}{3}(a^2 + b^2)$
Sphere of radius $a$ ....	Diameter	$\frac{2}{5}a^2$

## II. Kinetic energy ( $T$ ) and angular momentum ( $h$ ).

(a) Rigid body turning about fixed axis:

$$(7.602) \quad T = \frac{1}{2}I\omega^2, \quad h = I\omega.$$

(b) Rigid body in general plane motion:

$$(7.603) \quad T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}I\omega^2, \quad h = I\omega \quad (h, I \text{ about mass center}).$$

## III. Motion of a rigid body.

(a) Rotation about fixed axis:

$$(7.604) \quad I\dot{\omega} = N \quad (\text{angular momentum}),$$

$$(7.605) \quad \frac{1}{2}I\omega^2 + V = E \quad (\text{energy}).$$

(b) Compound pendulum:

- (i) Exact equation of motion:

$$(7.606) \quad k^2\ddot{\theta} + ga \sin \theta = 0, \quad (k \text{ relative to axis}).$$

(ii) Periodic time for small oscillations:

$$(7.607) \quad \tau = \frac{2\pi k}{\sqrt{ga}}.$$

(iii) Equivalent simple pendulum:

$$(7.608) \quad l = \frac{k^2}{a}.$$

(c) General motion parallel to plane:

$$(7.609) \quad m\ddot{x} = X, \quad m\ddot{y} = Y, \quad I\dot{\omega} = N \quad (\text{momentum});$$

$$(7.610) \quad \frac{1}{2}mq^2 + \frac{1}{2}I\omega^2 + V = E \quad (\text{energy}).$$

#### IV. Normal modes of vibration.

(a) A vibration is in general not periodic; it is composed of simple harmonic vibrations with different frequencies. These are the normal modes. A system vibrates in a normal mode if started under special initial conditions.

(b) The normal frequencies are found by assuming a simple harmonic solution of the equations of motion and solving a determinantal equation obtained on this assumption.

#### V. Stability of equilibrium.

Necessary and sufficient condition for stability: the potential energy is a minimum.

### EXERCISES VII

1. A uniform rod of length  $l$  and mass  $M$  is free to rotate in a vertical plane about an axis at a distance  $a$  from its center. If it is released from a horizontal position, find its angular velocity when passing through the vertical position.

2. A bucket of mass  $M$  is fastened to one end of a light rope; the rope is coiled round a windlass in the form of a circular cylinder (radius  $a$ ) which is left free to rotate about its axis. Prove that the bucket descends with acceleration

$$\frac{g}{1 + (I/Ma^2)},$$

where  $I$  is the moment of inertia of the cylinder about its axis.

3. Three uniform rods, each of mass  $m$ , form an equilateral triangle of side  $2a$ . The triangle is suspended from one corner. Find the lengths of the equivalent simple pendulums for oscillations under gravity

(i) when the triangle oscillates in its own plane;

(ii) when the plane of oscillation is perpendicular to the plane of the triangle.

4. A wheel consists of a thin rim of mass  $M$  and  $n$  spokes each of mass  $m$ , which may be considered as thin rods terminating at the center of the wheel. If the wheel is rolling with linear velocity  $v$ , express its kinetic energy in terms of  $M$ ,  $m$ ,  $n$ ,  $v$ .

With what acceleration will it roll down a rough inclined plane of inclination  $\alpha$ ?

5. A buoy is formed by joining the edge of a thin metal conical shell to the edge of a hemispherical shell of the same material and thickness. The radii of the hemisphere and of the mouth of the cone are each equal to 5 ft., and the slant height of the cone is 10 ft. The buoy is placed with the hemisphere in contact with the rough horizontal surface of a dock so that the axis is vertical. If slightly disturbed, determine whether or not it will return to the vertical position.

6. One end of a heavy chain is attached to a drum and the chain is wrapped around the drum, making  $n$  complete turns, with a small piece of chain hanging free. The drum is mounted on a smooth horizontal axle, and the chain is allowed to unwrap itself. Apply the principle of energy to find the angular velocity of the drum at the instant when the chain is completely unwrapped, in terms of the mass of the chain ( $m$ ), the radius of the drum ( $r$ ), and the moment of inertia of the drum ( $I$ ).

7. A rectangular plate swings in a vertical plane about one of its corners. If its period is 1 sec., find the length of the diagonal.

8. A particle of mass  $m$  moves in a plane under the action of a force with components

$$X = -k^2(2x + y), \quad Y = -k^2(x + 2y),$$

where  $k$  is a constant. What is the potential energy? Find the normal periods of oscillation about the position of equilibrium.

9. A uniform circular plate of radius  $a$  and mass  $M$  is dragged along a smooth sheet of ice by means of a long string attached to a point  $A$  on the rim of the plate. The tension  $T$  in the string is kept constant throughout. If initially the plate is at rest and the diameter through  $A$  makes a small angle with the string, show that this diameter oscillates about the direction of the string with a period equal to

$$2\pi \sqrt{\frac{Ma}{2T}}.$$

10. A particle  $A$  hangs from a fixed point by a light string, and another particle  $B$  of the same mass hangs from  $A$  by a second light string of the same length. Find the normal periods of oscillation, and sketch the normal modes

11. A homogeneous solid cylinder, whose section is a semicircle of radius  $a$ , rests with its flat face horizontal and in contact with a fixed rough circular cylinder of radius  $b$ , the generators of the two cylinders being parallel. Find the greatest value of  $a/b$  for which there is stability.

12. Find the radius of gyration of a uniform semicircular plate about a line through the mass center perpendicular to the plate.

13. A ladder (length  $2a$ ) rests against a smooth vertical wall and a smooth horizontal floor, the inclination to the floor being initially  $\alpha$ . Find the inclination of the ladder to the floor at the instant when the upper end leaves the wall as it slides down under the action of gravity.

14. A pendulum consists of a bob of mass  $m$  at the end of a light rod of length  $3a$ . It is suspended from the point of the rod distant  $2a$  from the bob. A horizontal force  $mb \cos nt$  (where  $b$  and  $n$  are constants and  $b$  is small) is applied to the rod at its upper end. Find the angular amplitude of the forced oscillations of period  $2\pi/n$ .

15. A uniform solid ellipsoid of revolution of semiaxes  $a$ ,  $b$  (the axis of revolution being  $2a$ ) is cut in two by a plane through the center perpendicular to the axis of revolution. If either half will balance in stable equilibrium with its vertex on a horizontal plane, prove that

$$\frac{b^2}{a^2} > \frac{1}{8}.$$

16. When a ship rolls through a small angle, the upward thrust of the water intersects the central plane of the ship at a point called the *metacenter*. Find a formula for the periodic time of rolling of a ship in terms of  $h$ , the height of the metacenter above the mass center of the ship, and  $k$ , the radius of gyration of the ship about a fore-and-aft axis through the mass center.

Is the periodic time increased or decreased by shifting cargo horizontally from the center of the ship to the sides, this shift being done symmetrically with respect to the central plane of the ship?

17. A square frame, consisting of four equal uniform rods of length  $2a$  rigidly joined together, hangs at rest in a vertical plane on two smooth pegs  $P$ ,  $Q$  at the same level. If  $PQ = c$  and the pegs are not both in contact with the same rod, show that there are three positions of equilibrium, provided  $a < c\sqrt{2}$ .

Of these positions, show that the only unstable one is the symmetrical position. If, however,  $a > c\sqrt{2}$ , show that the only possible position of equilibrium is stable.

18. A particle is suspended by a light string of length  $a$  from the lower end of a rod of the same mass and length  $2a$ , which is free to turn about its upper end. For vibrations about equilibrium in a vertical plane, show that the two normal frequencies are given by

$$\frac{1}{2\pi} \sqrt{\frac{gp}{a}},$$

where  $p$  satisfies

$$4p^2 - 25p + 9 = 0.$$

19. A rod of length  $2a$  hangs from a support which is given a small horizontal

displacement  $\xi$  varying with time according to the equation  $\xi = b \sin pt$ , where  $b$  and  $p$  are constants. Find the equation of motion for small oscillations. Integrate the equation, obtaining a result with two arbitrary constants. Find these constants on the assumption that when  $t = 0$  the rod is hanging vertically and has no angular velocity; hence show that the inclination  $\theta$  of the rod to the vertical is given by

$$\theta = \frac{bp^2n}{g(n^2 - p^2)} (n \sin pt - p \sin nt), \quad \left(n^2 = \frac{3g}{4a}\right).$$

**20.** Two simple pendulums, each of mass  $m$  and length  $a$ , hang from a trolley of mass  $M$  which can run without friction along horizontal rails. A small impulse, parallel to the rails, is applied to one of the pendulums and imparts to it an angular velocity  $\omega$ , the other pendulum and the trolley having no velocity at that instant. Investigate the resulting motion, and express the displacement of the trolley and the inclinations of the pendulums to the vertical as functions of the time.

Show that, if the ratio  $m/M$  is small, the motions of the pendulums relative to the trolley may be regarded as simple harmonic motions with slowly varying amplitudes, the amplitudes being given by the absolute values of

$$\frac{\omega}{p} \cos \frac{mpt}{2M}, \quad \frac{\omega}{p} \sin \frac{mpt}{2M},$$

where

$$p^2 = \frac{g}{a}.$$

\* More precisely, the moment of inertia about the line through the mass center perpendicular to the plane of motion.

\* In the particular case where  $V$  is constant (as for a sphere resting on a horizontal table), the equilibrium is often called *neutral*. Actually it is unstable in the sense of our definition.



## CHAPTER VIII

### PLANE IMPULSIVE MOTION

#### 8.1. GENERAL THEORY OF PLANE IMPULSIVE MOTION

##### The concept of an impulsive force.

For a particle moving in a plane under the action of a force with components  $X$ ,  $Y$ , the equations of motion are

$$(8.101) \quad m\ddot{x} = X, \quad m\ddot{y} = Y.$$

Multiplying by  $dt$  and integrating from  $t_0$  to  $t_1$ , we obtain

$$(8.102) \quad \Delta(m\dot{x}) = \int_{t_0}^{t_1} X dt, \quad \Delta(m\dot{y}) = \int_{t_0}^{t_1} Y dt,$$

where  $\Delta$  denotes an increment during the time interval  $(t_0, t_1)$ .

The vector with components

$$(8.103) \quad \int_{t_0}^{t_1} X dt, \quad \int_{t_0}^{t_1} Y dt$$

is called the *impulse* on the particle during the time interval  $(t_0, t_1)$ . We may state (8.102) in words as follows: *The increment in momentum is equal to the impulse.*

Let us now suppose that a particle of mass  $m$  can move along the  $x$ -axis. At time  $t = 0$ , it is at rest at  $x = 0$ . At this instant a force

$$(8.104) \quad X = A \sin \frac{\pi t}{\tau}$$

commences to act and acts until  $t = \tau$ . ( $A$  and  $\tau$  are constants.) During this time the equation of motion of the particle is

$$(8.105) \quad m\ddot{x} = A \sin \frac{\pi t}{\tau},$$

and so

$$u = \dot{x} = \frac{A\tau}{\pi m} \left( 1 - \cos \frac{\pi t}{\tau} \right),$$

$$x = \frac{A\tau t}{\pi m} - \frac{A\tau^2}{\pi^2 m} \sin \frac{\pi t}{\tau},$$

the constants of integration having been chosen to fit the initial conditions. Thus in the time interval  $(0, \tau)$  the particle receives increments in velocity and position given by

$$(8.106) \quad \Delta u = \frac{2A\tau}{\pi m}, \quad \Delta x = \frac{A\tau^2}{\pi m}.$$

The mean value of the force is

$$(8.107) \quad \bar{X} = \frac{1}{\tau} \int_0^\tau A \sin \frac{\pi t}{\tau} dt = \frac{2A}{\pi},$$

and so (8.106) may be written

$$(8.108) \quad \Delta u = \frac{\bar{X}\tau}{m}, \quad \Delta x = \frac{1}{2} \frac{\bar{X}\tau^2}{m}.$$

The experiment may be repeated using different values of  $A$  and  $\tau$ . We note that as long as the product  $\bar{X}\tau$  remains unchanged the value of  $\Delta u$  remains the same. If we let  $A$  (and therefore  $\bar{X}$ ) tend to infinity and let  $\tau$  tend to zero in such a way that  $\bar{X}\tau$  maintains a fixed value  $C$ , we have

$$(8.109) \quad \Delta u \rightarrow \frac{C}{m}, \quad \Delta x \rightarrow 0.$$

We note that, in this limiting case of “an infinite force acting for an infinitesimal time,” there is an instantaneous change in velocity but no change in position.

Returning to the general equations (8.102), we may let the force components  $(X, Y)$  tend to infinity and the time interval  $t_1 - t_0$  to zero in such a way that the integrals remain constant or approach finite limits. Under these circumstances a particle moving in a plane experiences (in the limit) an instantaneous change of velocity. Since the velocity remains finite during this change, the displacement is zero in the limit. The instantaneous change in momentum is given by

$$(8.110) \quad \Delta(m\dot{x}) = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} X dt, \quad \Delta(m\dot{y}) = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} Y dt.$$

Have we here introduced a new ingredient or concept into mechanics? It must be admitted that we have, because no force, however large, can produce an *instantaneous* change in momentum. To place our new ideas on a secure foundation, we admit the concept of an *impulsive force*, with components denoted by  $\hat{X}$ ,  $\hat{Y}$ : it is such that, when applied to a particle, the impulsive force causes an instantaneous change in momentum given by

$$(8.111) \quad \Delta(m\dot{x}) = \hat{X}, \quad \Delta(m\dot{y}) = \hat{Y}.$$

We must, however, regard the impulsive force, not as something absolutely new, but as connected with the ordinary force ( $X$ ,  $Y$ ) by the relations

$$(8.112) \quad \hat{X} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} X dt, \quad \hat{Y} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} Y dt,$$

obtained by comparison of (8.110) and (8.111).

On account of this connection, it is unnecessary to repeat for impulsive forces results already obtained for ordinary forces. We draw attention to the fact that impulsive reactions between the particles in a rigid body obey the law of action and reaction. The theory of moments applies to impulsive forces, and we may speak of an impulsive couple. The idea of equipollence may also be used.

It is clear from (8.112) that forces which remain finite as  $t_1 \rightarrow t_0$  (e.g., gravity) contribute nothing to the impulsive force.

An impulsive force is the product of an ordinary force and a time and is equal to a change in momentum. Hence, impulsive force has the dimensions  $[MLT^{-1}]$ ; its magnitude is expressed in dyne sec. or gm. cm. sec.<sup>-1</sup> in the e.g.s. system and in poundal sec. or lb. ft. sec.<sup>-1</sup> in the f.p.s. system.

## Principles of linear and angular momentum.

We expressed in (5.206) the law that, for any system, the rate of change of linear momentum is equal to the sum of the external forces. If  $M_x$ ,  $M_y$  are the components of linear momentum in the directions of axes  $O_x$ ,  $O_y$ , and  $X$ ,  $Y$  the total components of external force in those directions, then

$$(8.113) \quad \dot{M}_x = X, \quad \dot{M}_y = Y.$$

Let us multiply by  $dt$ , integrate from  $t = t_0$  to  $t = t_1$ , and then proceed to the limit  $t_1 \rightarrow t_0$ , supposing the forces to tend to infinity. Then

$$(8.114) \quad \Delta M_x = \hat{X}, \quad \Delta M_y = \hat{Y},$$

where  $\hat{X}$ ,  $\hat{Y}$  are the sums of the components of the external impulsive forces. In words, *the sudden change in the linear momentum of a system is equal to the total external impulsive force.*

These equations may also be written in vector form:

$$(8.115) \quad \Delta \mathbf{M} = \hat{\mathbf{F}},$$

where  $\hat{\mathbf{F}}$  is the vector sum of the external impulsive forces. Similarly, we obtain from (5.209) the vector equation

$$(8.116) \quad m\Delta\bar{\mathbf{q}} = \hat{\mathbf{F}},$$

where  $m$  is the mass of the system and  $\Delta\bar{\mathbf{q}}$  the sudden change in the velocity of the mass center.

If there are no external impulsive forces, we have  $\hat{\mathbf{F}} = 0$ , and hence  $\Delta\bar{\mathbf{q}} = 0$ . Thus, when the impulsive forces are purely internal, there is no sudden change in the velocity of the mass center.

This result is of interest in connection with collisions and explosions. Here, in physical reality, we find large forces acting for short intervals of time, and we may treat the phenomena mathematically by means of impulsive forces. Thus, if a shunting locomotive strikes a car, the mass center of the system “locomotive + car” has the same velocity just before and just after the collision. The bursting of a shell in the air produces a set of fragments, the mass center of which has the same velocity as the mass center of the shell before bursting.

Consider now the change in angular momentum about a fixed line due to the action of impulsive forces. Equation (5.214) applies. Multiplying by  $dt$ , integrating over the range  $(t_0, t_1)$  and proceeding to the limit as usual, we find

$$(8.117) \quad \Delta h = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} N dt = \hat{N}.$$

In words, *the sudden change in angular momentum about the fixed axis is equal to the moment of the external impulsive forces about the axis.*\*

We may treat similarly the equation (5.219) which concerns motion relative to the mass center. We find that *the sudden change in angular momentum relative to the mass center is equal to the moment of the external impulsive forces about the mass center.*

If the system is a rigid body with a fixed axis, (8.117) may be written

$$(8.118) \quad I \Delta\omega = \hat{N},$$

where  $I$  is the moment of inertia about the fixed axis,  $\Delta\omega$  the change in angular velocity, and  $\hat{N}$  the moment of the impulsive forces about the axis.

For a rigid body which can move parallel to a plane, (8.118) holds, provided we understand  $I$  to be the moment of inertia about the mass center and  $\hat{N}$  the moment of impulsive forces about the mass center.

We have now converted the principles of linear and angular momentum into forms valid in the case where impulsive forces act. Since the application of impulsive forces leads to sudden changes in velocity, we may call the theory of impulsive motion a *discontinuous* theory, reserving the word *continuous* for those cases in which no sudden changes in velocity occur.

In the continuous theory the principle of energy is useful for determining motions, either completely or in part. But it must be used with great caution in the discontinuous theory, because we find in general that when impulsive forces act the law of conservation of mechanical energy does not hold. Actually, the energy is not lost; it is converted into heat or employed to deform the bodies on which the impulses act. But *mechanical* energy disappears, and it is with mechanical energy alone that we are concerned in this book.

*Exercise.* An impulsive force  $\hat{\mathbf{P}}$  is applied at one end of a bar of mass  $m$  and length  $2a$ , in a direction perpendicular to the bar. Find the velocity imparted to the other end of the bar, assuming (i) that the center of the bar is fixed, (ii) that the bar is free.

## 8.2. COLLISIONS

As remarked above, a collision between two bodies gives rise (in physical reality) to large reactions acting for a short time, and so we treat the problem of collision mathematically by means of impulsive forces.

### The collision of spheres and the coefficient of restitution.

As an illustrative example, we shall discuss the problem of the collision of two spheres which are moving along the line joining their centers (Fig. 97).

Taking the axis  $Ox$  along the line of centers, let us use the following notation:

$m_1, m_2$  = masses of the spheres,  
 $u_1, u_2$  = velocities of centers before collision,  
 $u'_1, u'_2$  = velocities of centers after collision,  
 $\hat{P}$  = magnitude of impulsive reaction.

We have then

$$(8.201) \quad m_1(u'_1 - u_1) = -\hat{P}, \quad m_2(u'_2 - u_2) = \hat{P},$$

and so

$$(8.202) \quad m_1u'_1 + m_2u'_2 = m_1u_1 + m_2u_2,$$

as indeed we might have deduced directly from the fact that there is no external impulsive force.

Our problem is to find the result of the collision, i.e., to find  $u'_1, u'_2$  when  $u_1, u_2$  are given. But for this we have only one equation (8.202), and that is not enough to give two unknowns. We can proceed no further without an additional hypothesis, and here we introduce the idea of the *coefficient of restitution*.

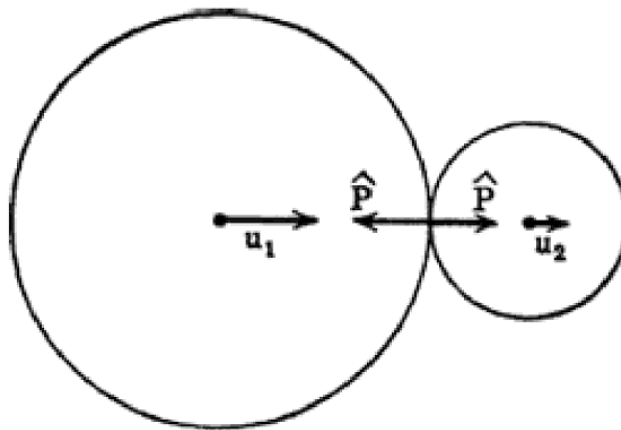


FIG. 97.—Collision of two spheres.

Consider the problem of collision as it might occur in reality, say between two tennis balls. Actually the balls would become distorted during the collision and then would bound away from one another, regaining their spherical shapes. This is a complicated process which we cannot follow through mathematically, and we are obliged to substitute some simple hypothesis based on experimental results.

We introduce the expressions *speed of approach*  $q_a$  and *speed of separation*  $q_s$ . For a general collision, these speeds are calculated for the particles of the two bodies at the point of contact, components of velocity along the common normal at that point being used. In our problem of colliding spheres, we have

$$(8.203) \quad q_a = u_1 - u_2, \quad q_s = u'_2 - u'_1.$$

The following general hypothesis is adopted: *The speeds of separation and*

approach are connected by the relation

$$(8.204) \quad q_s = eq_a,$$

where  $e$  is a positive number, called the *coefficient of restitution*. The value of  $e$  depends on the materials of which the bodies are composed and also on their shapes and sizes; it never exceeds unity in value. When  $e = 1$  the bodies are said to be *perfectly elastic*, and when  $e = 0$  they are said to be *perfectly inelastic*.

In the problem of the spheres, we now have

$$(8.205) \quad \begin{cases} m_1 u'_1 + m_2 u'_2 = m_1 u_1 + m_2 u_2, \\ u'_2 - u'_1 = e(u_1 - u_2). \end{cases}$$

Hence,

$$(8.206) \quad \begin{cases} u'_1 = \frac{m_1 - em_2}{m_1 + m_2} u_1 + (1 + e) \frac{m_2}{m_1 + m_2} u_2, \\ u'_2 = (1 + e) \frac{m_1}{m_1 + m_2} u_1 + \frac{m_2 - em_1}{m_1 + m_2} u_2, \end{cases}$$

and so the problem is solved.

In the case of perfectly inelastic spheres ( $e = 0$ ), we have  $u'_1 = u'_2$ ; there is no rebound.

If the spheres are of the same mass ( $m_1 = m_2$ ) and there is perfect elasticity ( $e = 1$ ), we have

$$(8.207) \quad u'_1 = u_2, \quad u'_2 = u_1;$$

this means that the spheres exchange velocities. This case is particularly interesting because, in the kinetic theory of gases, the mathematical model represents the molecules by perfectly elastic spheres.

### Compression and restitution.

The hypothesis (8.204) appears artificial; we generally prefer to adopt hypotheses which have some plausibility. The hypothesis may however be put in another form, which suggests rather better its connection with physical reality. To do this, we return to the physical picture of the collision of two tennis balls. At first the centers of the balls are approaching one another, and the balls are being distorted. Then they start to regain their spherical shapes, pressing against one another until they separate. Thus the whole period of collision is divided into a period of *compression* and a period of *restitution*. We



may adopt, instead of (8.204), the following hypothesis: The impulse during restitution bears to the impulse during compression a definite ratio  $e$ . Or, passing to the limit of infinite forces and vanishing time, we may make the following formal statement of our hypothesis: *If  $\hat{P}_1$  is the magnitude of the impulsive reaction of compression required to reduce the speed of approach to zero, then the magnitude  $\hat{P}_2$  of the impulsive reaction of restitution is*

$$(8.208) \quad \hat{P}_2 = e\hat{P}_1$$

where  $e$  is the coefficient of restitution. These two impulsive reactions act in the same direction.

It is by no means obvious that (8.208) is equivalent to (8.204), but it can be proved without much difficulty. We shall here merely establish the equivalence for the problem of the spheres. We have, for compression,

$$(8.209) \quad m_1u - m_1u_1 = -\hat{P}_1, \quad m_2u - m_2u_2 = \hat{P}_1,$$

where  $u$  is the common velocity when the speed of approach is zero. For restitution, we have by (8.208)

$$(8.210) \quad m_1u'_1 - m_1u = -e\hat{P}_1, \quad m_2u'_2 - m_2u = e\hat{P}_1.$$

We have here four equations, which can be solved for  $u'_1, u'_2, u, \hat{P}_1$ . To prove that we get the same result as that given by (8.204), we eliminate  $u$  from (8.209) and also from (8.210); we obtain

$$(8.211) \quad \begin{cases} m_1m_2(u_1 - u_2) = (m_1 + m_2)\hat{P}_1, \\ m_1m_2(u'_2 - u'_1) = e(m_1 + m_2)\hat{P}_1, \end{cases}$$

from which (8.204) follows at once.

### **Motion relative to the mass center.**

The mathematics of discontinuous motions is much simpler than that of continuous motions, because the equations to be solved are algebraic, not differential. But the algebra may become complicated, and it is sometimes advisable to use a special Newtonian frame of reference. Thus, in the case of the two spheres considered above, we may use a frame of reference in which the mass center is at rest before collision. It is, of course, at rest in this frame after collision also. We have then

$$(8.212) \quad \begin{cases} m_1 u_1 + m_2 u_2 = 0, & m_1 u'_1 + m_2 u'_2 = 0, \\ u'_2 - u'_1 = e(u_1 - u_2); \end{cases}$$

hence,

$$(8.213) \quad u'_1 = -eu_1, \quad u'_2 = -eu_2.$$

As a result of the collision, the components of velocity are reversed in sign and multiplied by the coefficient of restitution.

If the kinetic energy is  $T$  before collision and  $T'$  after collision, we have

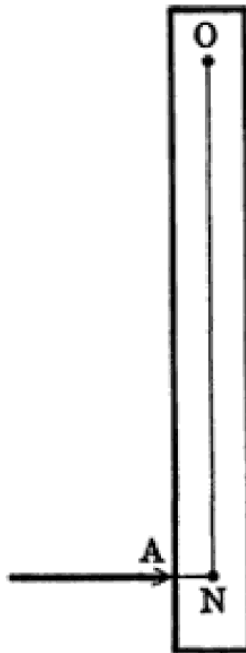
$$T = \frac{1}{2}(m_1 u_1^2 + m_2 u_2^2), \quad T' = \frac{1}{2}(m_1 u'^2_1 + m_2 u'^2_2) = e^2 T,$$

and so the loss of kinetic energy is

$$(8.214) \quad T - T' = (1 - e^2)T.$$

Since  $e \leq 1$ , kinetic energy is lost in every case except that of perfect elasticity ( $e = 1$ ).

We have discussed the collision of spheres in the case where their centers move along the line joining the centers. Provided the spheres are smooth, the extension to the case where the spheres have general motions is immediate; the components of momentum (and hence velocity) in directions parallel to the common tangent plane of the spheres undergo no changes, and the components of velocity along the common normal change as described above.



### 8.3. APPLICATIONS

We shall now illustrate the application of the principles of linear and angular momentum by two examples.

### The ballistic pendulum.

Consider a rigid body, hanging in equilibrium from a horizontal axis  $O$  (Fig. 98). A bullet, traveling horizontally, strikes the body at  $A$  and becomes embedded in it. As a result of the impact, the body swings as a compound pendulum, rising through an angular displacement  $\alpha$  before coming to rest. On account of its importance in ballistics, the apparatus is called a *ballistic pendulum*. From the angle  $\alpha$  and the constants of the system, we can compute the velocity of the bullet, as we shall now show.

FIG. 98.—Ballistic pendulum.

Let us take as dynamical system the body and the bullet. Then the forces between the body and the bullet are internal. During the brief interval of impact the only external forces acting are (i) gravity and (ii) the reaction at  $O$ . The force of gravity is a finite force and so contributes no impulsive force. Since the reaction at  $O$  has no moment about  $O$ , it is evident that the principle of angular momentum enables us to state that

angular momentum of system about  $O$  before impact

= angular momentum of system about  $O$  after impact.

Let  $ON$  be the vertical through  $O$ ,  $AN$  being horizontal. Let  $m$  be the mass of the bullet,  $q$  its speed,  $I$  the moment of inertia of the body about  $O$ , and  $\omega$  the angular velocity immediately after impact. Then the angular momentum of the system about  $O$  is

before impact:  $mq \cdot ON$ ,

after impact:  $(m \cdot AO^2 + I)\omega$ .

If the mass of the bullet is very small in comparison with that of the body, we may neglect  $m \cdot AO^2$  in comparison with  $I$ ; thus we have

$$(8.301) \quad mql = I\omega,$$

where  $l = ON$ .

Although we cannot apply the principle of energy during impact, we can apply it in the subsequent motion. Thus, again neglecting the mass of the bullet in comparison with that of the body, we have

$$(8.302) \quad \frac{1}{2}I\omega^2 = Mgh(1 - \cos \alpha),$$

where  $M$  is the mass of the body and  $h$  the distance of its mass center from  $O$ .

Hence, from (8.301),

$$(8.303) \quad q = \frac{\sqrt{2IMgh(1 - \cos \alpha)}}{ml},$$

which gives the speed of the bullet in terms of  $\alpha$  and constants of the system.

### Linked rods.

Two uniform rods  $AB$ ,  $BC$ , each of mass  $m$  and length  $2a$ , are connected by a smooth joint at  $B$  and lie in one straight line on a smooth horizontal table (Fig. 99a). A horizontal blow  $\hat{\mathbf{P}}$  is struck at  $C$ , in a direction perpendicular to  $BC$ . We wish to find the motion generated.

Let us draw a schematic diagram (Fig. 99b), separating the rods in order to represent the reactions without confusion.

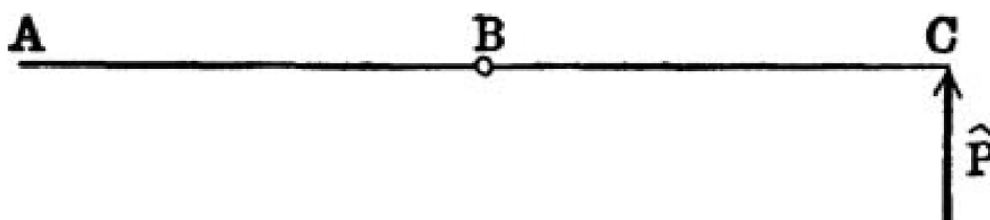


FIG. 99a.—A pair of rods, linked at  $B$ , receive a blow at  $C$ .

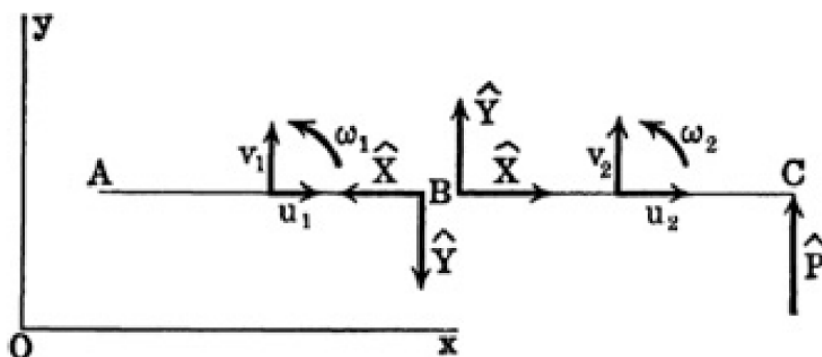


FIG. 99b.—Diagram of velocities and impulsive forces.

Taking rectangular axes  $Oxy$ , with  $Ox$  parallel to  $ABC$  and  $Oy$  in the sense of  $\hat{\mathbf{P}}$ , we shall use the following notation:

- $u_1, v_1$  = components of velocity of center of  $AB$ ,
- $u_2, v_2$  = components of velocity of center of  $BC$ ,
- $\omega_1$  = angular velocity of  $AB$ ,
- $\omega_2$  = angular velocity of  $BC$ ,
- $\hat{X}, \hat{Y}$  = components of reaction on  $BC$  at  $B$ ,
- $-\hat{X}, -\hat{Y}$  = components of reaction on  $AB$  at  $B$ .

Since the rods are joined at  $B$ , this point must have the same velocity whether considered as a point of  $AB$  or of  $BC$ . Thus,

$$(8.304) \quad u_1 = u_2, \quad v_1 + a\omega_1 = v_2 - a\omega_2.$$

The principle of linear momentum applied to each rod gives

$$(8.305) \quad \begin{cases} mu_1 = -\hat{X}, & mu_2 = \hat{X}, \\ mv_1 = -\hat{Y}, & mv_2 = \hat{Y} + \hat{P}; \end{cases}$$

the principle of angular momentum gives

$$(8.306) \quad mk^2\omega_1 = -a\hat{Y}, \quad mk^2\omega_2 = -a\hat{Y} + a\hat{P},$$

where  $k$  is the radius of gyration of each rod about its center, so that  $k^2 = \frac{1}{3}a^2$ .

From the first equation of (8.304) and the first two of (8.305), we see that

$$(8.307) \quad u_1 = u_2 = 0, \quad \hat{X} = 0.$$

There now remain in (8.304), (8.305), and (8.306) five equations for the following five unknowns:

$$v_1, v_2, \omega_1, \omega_2, \hat{Y}.$$

It is most symmetrical to find  $\hat{Y}$  first by substitution in (8.304) from the other equations; we find

$$(8.308) \quad \hat{Y} = \frac{1}{4}\hat{P},$$

and hence

$$(8.309) \quad \begin{cases} v_1 = -\frac{1}{4}\frac{\hat{P}}{m}, & v_2 = \frac{5}{4}\frac{\hat{P}}{m}, \\ \omega_1 = -\frac{3}{4}\frac{\hat{P}}{ma}, & \omega_2 = \frac{9}{4}\frac{\hat{P}}{ma}. \end{cases}$$

The velocity of  $B$  is  $\hat{P}/m$ , downward in the diagram.

## 8.4. SUMMARY OF PLANE IMPULSIVE MOTION

### I. Components of impulsive force.

$$(8.401) \quad \hat{X} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} X \, dt, \quad \hat{Y} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} Y \, dt.$$

## II. Instantaneous change in motion.

(a) Particle:

$$(8.402) \quad \Delta(m\dot{x}) = \hat{X}, \quad \Delta(m\dot{y}) = \hat{Y}.$$

(b) Any system:

$$(8.403) \quad \Delta M_x = \hat{X}, \quad \Delta M_y = \hat{Y}, \quad \Delta h = \hat{N}.$$

( $\hat{X}, \hat{Y}, \hat{N}$  = total components and moment of external impulsive forces.)

(c) Rigid body with fixed axis:

$$(8.404) \quad I \Delta\omega = \hat{N}.$$

(d) Rigid body moving parallel to a fixed plane:

$$(8.405) \quad m \Delta u = \hat{X}, \quad m \Delta v = \hat{Y}, \quad I \Delta\omega = \hat{N}.$$

( $u, v$  = components of velocity of mass center;  $\hat{N}$  = impulsive moment about mass center.)

## III. Collisions.

Either

$$(8.406) \quad q_a = e q_a$$

or

$$(8.407) \quad \hat{P}_2 = e \hat{P}_1.$$

( $e$  = coefficient of restitution;  $e \leq 1$ .)

## EXERCISES VIII

1. A bar 2 ft. long, of mass 10 lb., lies on a smooth horizontal table. It is struck horizontally at a distance of 6 in. from one end, the blow being perpendicular to the bar; the magnitude of the blow is such that it would impart a velocity of 3 ft. per sec. to a mass of 2 lb. Find the velocities of the ends of the bar just after it is struck.

2. A uniform rod of mass  $m$  and length  $3a$  hangs from a pin passing through it at a distance  $a$  from the upper end. Find in terms of  $m, a, g$  the magnitude of the smallest blow, struck at the lower end of the rod, which will make the rod describe a complete revolution.

3. A ball is dropped on the floor from a height  $h$ . If the coefficient of restitution is  $e$ , find the height of the ball at the top of the  $n$ th rebound.

4. A bar, 6 ft. long, is swinging about a horizontal axle passing through it at a distance of 1 foot from one end. At what point must a blow be struck to bring it to rest without causing any impulsive reaction on the axle? (This point is called the *center of percussion*.)

5. A particle moving with a speed of 30 feet per second in a direction making an angle of  $60^\circ$  with the horizontal strikes a smooth horizontal plane and rebounds, the coefficient of restitution being  $\frac{1}{2}$ . Find the speed and the direction of motion of the particle immediately after impact.

6. A uniform square plate of mass  $M$  and side  $2a$  rests on a smooth horizontal table. A horizontal impulsive force of magnitude  $\hat{P}$  is applied at a corner in a direction perpendicular to the diagonal at that corner. Show that the angular velocity generated by this impulsive force is

$$\frac{3\sqrt{2}\hat{P}}{2Ma}.$$

7. A tug of mass  $m$  tons is attached to a barge of mass  $M$  tons by a cable the mass of which may be neglected. The cable is slack. The tug moves and has acquired a speed of  $v$  ft. per sec. when the cable becomes taut and the barge is jerked into motion. Assuming that the cable has a coefficient of restitution  $\frac{1}{2}$  and neglecting the impulsive resistance of the water, find

(i) the speed imparted to the barge;

(ii) the mean tension (in tons wt.) in the cable during the jerk, supposing this to take  $t$  sec.

8. A billiard ball of radius  $a$  and mass  $M$  rests on a horizontal table. In a vertical plane through the center of the ball there is applied a horizontal impulsive force of magnitude  $\hat{P}$ . If the line of action of the impulse is at a height  $h$  above the table, find the initial velocity of that point of the ball which is in contact with the table.

9. Two gear wheels of radii  $a_1, a_2$  and axial moments of inertia  $I_1, I_2$ , respectively, can rotate freely about fixed parallel axes. Initially the wheel of radius  $a_1$  is rotating with angular velocity  $\omega$ , while the other wheel is at rest. If the gear wheels are suddenly engaged, find the angular velocity of each wheel afterward.

10. A beam of mass 100 lb. and length 6 ft. hangs from an axle passing through it at a distance of 1 ft. from one end. It is drawn aside through an angle of  $30^\circ$  and then released. It is stopped dead at the lowest point of its swing by a horizontal blow which strikes it at a height of 2 ft. above its lower end. What is the impulsive reaction on the axle, expressed in lb. ft. sec.<sup>-1</sup>?

11. Two uniform rods  $AB, BC$ , each of mass  $M$  and length  $2a$ , are smoothly jointed together and rest on a smooth horizontal plane, the angle between the rods being  $45^\circ$ . A horizontal impulsive force  $\hat{P}$  is applied at  $A$  in a direction at right angles to  $AB$  and away from the rod  $BC$ . Find the initial angular velocity of  $BC$ .



## CHAPTER IX

# PRODUCTS OF VECTORS

Up to the present, our development of mechanics has been restricted, for the most part, to two dimensions. We now come to the systematic treatment of mechanics in space. Here we must make a decision as to notation. On the one hand, we have the ordinary notation of coordinates; on the other hand, the vector symbolism. Each has its advantages, but on the whole the vector notation has proved more useful on account of its compactness. We shall therefore use it extensively (but not exclusively) throughout the rest of the book. The present chapter, together with Sec. 1.3, explains the mathematical language to be employed later.

### 9.1. THE SCALAR AND VECTOR PRODUCTS

In developing the theory of vectors, we try to extend to vectors the operations of ordinary (scalar) algebra, as far as possible. In Sec. 1.3, this was done successfully for the addition and subtraction of vectors and for the multiplication of a vector by a scalar. We now consider the multiplication of vectors by one another, and here the methods of ordinary algebra are not so easy to generalize. Actually, we define two types of product—the *scalar product* and the *vector product*.

As in Sec. 1.3, we use  $P_1, P_2, P_3$  to denote the components of a vector  $\mathbf{P}$  on rectangular axes  $Ox, Oy, Oz$ , and  $P$  to denote its magnitude.

#### Scalar product.

The scalar product of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$ , written  $\mathbf{P} \cdot \mathbf{Q}$ , is defined by

$$(9.101) \quad \mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{P}$  and  $\mathbf{Q}$ . Since  $Q \cos \theta$  is the component of  $\mathbf{Q}$  in the direction of  $\mathbf{P}$  (cf. Sec. 1.3), it is clear that  $\mathbf{P} \cdot \mathbf{Q}$  is equal to the magnitude of  $\mathbf{P}$  multiplied by the component of  $\mathbf{Q}$  in the direction of  $\mathbf{P}$ .

*Exercise.* A vector has components  $(1, 3, -2)$  in the directions of rectangular axes  $Oxyz$ . What is its component along the line  $x = y = z$ , the positive sense being that in which  $x$  increases?

## Positive rotations.

Before defining the vector product, we shall introduce a convention concerning the sign of a rotation.

A rotation about a directed line  $L$  is said to be *positive* if it bears to the direction of  $L$  the same relation as the rotation of a right-handed screw bears to its direction of travel (Fig. 100). Thus a rotation from south to east is a positive rotation about the upward vertical; the earth's rotation about its axis drawn from south to north is also positive.

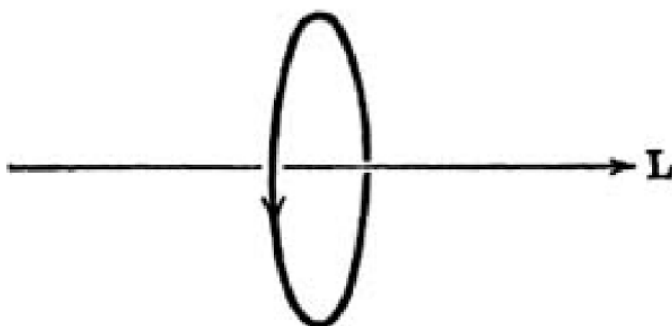


FIG. 100.—A positive rotation.

## Right-handed triads.

Consider three non-coplanar vectors. These three vectors, taken in some order, form an *ordered triad*. Since all the triads of which we shall speak are ordered, the adjective will be understood in future, and a *triad* will mean an ordered triad. Let  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  be an orthogonal triad, the order being as indicated. This triad is said to be *right-handed* if the rotation through a right angle from  $\mathbf{P}$  to  $\mathbf{Q}$  is a positive rotation about  $\mathbf{R}$ . Any other triad is said to be right-handed if it can be deformed continuously into a right-handed orthogonal triad without its vectors becoming coplanar at any stage in the deformation.

If the triad  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  is right-handed, then the triad  $\mathbf{Q}$ ,  $\mathbf{P}$ ,  $\mathbf{R}$  is said to be *left-handed*.

If the triad of unit coordinate vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , introduced in Sec. 1.3, is a right-handed triad, the axes  $Oxyz$  are said to be right-handed. We shall always use right-handed axes for the sake of consistency.

## Vector product.

Given two vectors  $\mathbf{P}$  and  $\mathbf{Q}$ , we draw the unit vector  $\mathbf{n}$  perpendicular to both  $\mathbf{P}$  and  $\mathbf{Q}$ , such that the triad  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{n}$  is a right-handed triad. We define the vector product of  $\mathbf{P}$  and  $\mathbf{Q}$ , written  $\mathbf{P} \times \mathbf{Q}$ , by

$$(9.103) \quad \mathbf{P} \times \mathbf{Q} = PQ \sin \theta \mathbf{n},$$

where  $\theta$  is the angle between  $\mathbf{P}$  and  $\mathbf{Q}$  (Fig. 101).

It is clear from the definition that a rotation from  $\mathbf{P}$  to  $\mathbf{Q}$ , through an angle less than two right angles, is a positive rotation about  $\mathbf{P} \times \mathbf{Q}$ . We note that the magnitude of  $\mathbf{P} \times \mathbf{Q}$  is  $PQ \sin \theta$ ; this is equal to the area of the parallelogram whose adjacent sides are  $\mathbf{P}$  and  $\mathbf{Q}$ .

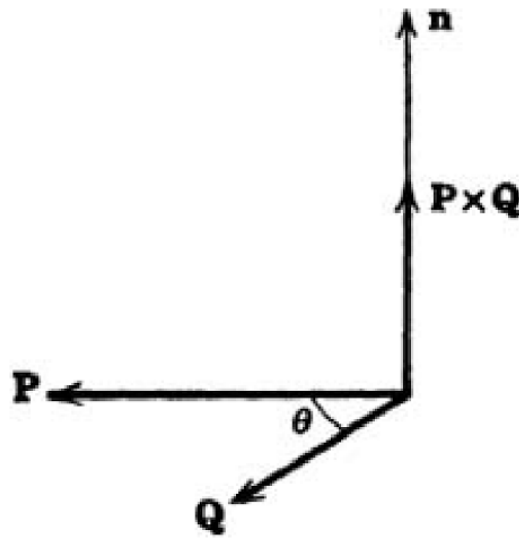


FIG. 101.—The vector product.

The vector product  $\mathbf{P} \times \mathbf{Q}$  of two non-zero vectors vanishes if, and only if,  $\mathbf{P}$  and  $\mathbf{Q}$  are codirectional or opposite; in particular,

$$\mathbf{P} \times \mathbf{P} = \mathbf{0}.$$

Let us now find the components of  $\mathbf{P} \times \mathbf{Q}$ . If we denote this vector by  $\mathbf{R}$ , then  $\mathbf{R}$  is perpendicular to both  $\mathbf{P}$  and  $\mathbf{Q}$ , and we have

$$\begin{aligned} R_1 P_1 + R_2 P_2 + R_3 P_3 &= 0, \\ R_1 Q_1 + R_2 Q_2 + R_3 Q_3 &= 0. \end{aligned}$$

Therefore,

$$(9.104) \quad \begin{cases} R_1 = k(P_2 Q_3 - P_3 Q_2), \\ R_2 = k(P_3 Q_1 - P_1 Q_3), \\ R_3 = k(P_1 Q_2 - P_2 Q_1), \end{cases}$$

where  $k$  is an undetermined factor. Now,

$$\begin{aligned}
 R^2 &= R_1^2 + R_2^2 + R_3^2 \\
 &= k^2[(P_2Q_3 - P_3Q_2)^2 + (P_3Q_1 - P_1Q_3)^2 + (P_1Q_2 - P_2Q_1)^2] \\
 &= k^2[(P_1^2 + P_2^2 + P_3^2)(Q_1^2 + Q_2^2 + Q_3^2) \\
 &\quad - (P_1Q_1 + P_2Q_2 + P_3Q_3)^2] \\
 &= k^2P^2Q^2 \sin^2 \theta.
 \end{aligned}$$

But, by definition,

$$R = PQ \sin \theta,$$

and hence

$$k = \pm 1.$$

From considerations of continuity, it is evident that  $k$  is to have the same sign in all cases. This sign may therefore be determined by considering the particular case where  $\mathbf{P}$  and  $\mathbf{Q}$  are unit vectors directed along the positive axes of  $x$  and  $y$ , respectively; then,

$$\begin{aligned}
 P_1 &= 1, P_2 = 0, P_3 = 0, \\
 Q_1 &= 0, Q_2 = 1, Q_3 = 0,
 \end{aligned}$$

and hence, by (9.104),

$$R_1 = 0, R_2 = 0, R_3 = k.$$

But  $\mathbf{P} \times \mathbf{Q}$  is, in this case, a unit vector directed along the positive axis of  $z$ , so that  $R_3 = 1$ . Hence,  $k = +1$  here, and so in all cases. Thus, quite generally, the components of  $\mathbf{R} = \mathbf{P} \times \mathbf{Q}$  are

$$(9.105) \quad \begin{cases} R_1 = P_2Q_3 - P_3Q_2, \\ R_2 = P_3Q_1 - P_1Q_3, \\ R_3 = P_1Q_2 - P_2Q_1. \end{cases}$$

Note that the number describing the component and the subscripts in the leading term of the expression for that component are a cyclic permutation of the numbers 1, 2, 3.

From the definition (9.103), it is evident that

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \mathbf{R} \cdot (\mathbf{P} \times \mathbf{Q}),$$

and

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = -\mathbf{P} \cdot (\mathbf{R} \times \mathbf{Q}) = -\mathbf{Q} \cdot (\mathbf{P} \times \mathbf{R}).$$

Thus a mixed triple product is not changed by a cyclic permutation of the vectors; its sign is reversed when two of the vectors are interchanged.

We may interpret the mixed triple product geometrically as follows: As we have seen, the magnitude of  $\mathbf{Q} \times \mathbf{R}$  is equal to the area of the parallelogram whose adjacent sides represent  $\mathbf{Q}$  and  $\mathbf{R}$ . Now  $\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R})$  is the product of the magnitude of  $\mathbf{Q} \times \mathbf{R}$  by the component of  $\mathbf{P}$  in the direction of  $\mathbf{Q} \times \mathbf{R}$  (that is, perpendicular to the plane of  $\mathbf{Q}$  and  $\mathbf{R}$ ). Hence the *magnitude* of  $\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R})$  is equal to the volume of the parallelepiped whose adjacent edges represent  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$  (Fig. 102). The *sign* of  $\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R})$  is also significant; it is positive or negative according as the angle between  $\mathbf{P}$  and  $\mathbf{Q} \times \mathbf{R}$  is acute or obtuse, i.e., according as  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  form a right- or left-handed triad.

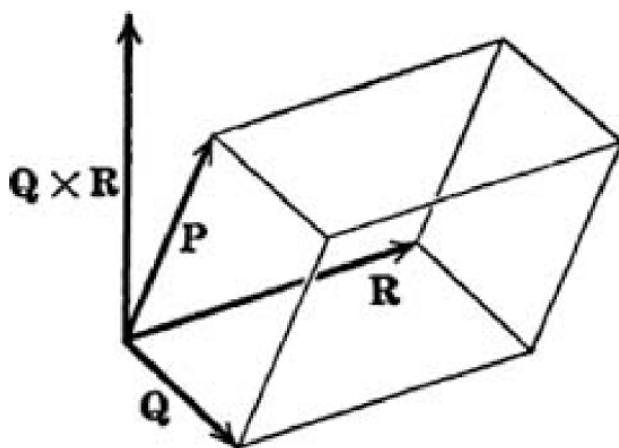


FIG. 102.—The mixed triple product.

From the geometrical interpretation, it is obvious that

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = 0$$

if the vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  are coplanar.

### Vector triple product.

From the vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ , we can form another product, namely,  $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$ ; this is evidently a vector and is called the *vector triple product*.

We shall now express this product as the difference of two vectors. Denoting it by  $\mathbf{U}$  and writing  $\mathbf{V} = \mathbf{Q} \times \mathbf{R}$ , we have

$$\mathbf{U} = \mathbf{P} \times \mathbf{V};$$

hence, using (9.105),

$$\begin{aligned} U_1 &= P_2 V_3 - P_3 V_2 \\ &= P_2(Q_1 R_2 - Q_2 R_1) - P_3(Q_3 R_1 - Q_1 R_3) \\ &= (P_1 R_1 + P_2 R_2 + P_3 R_3)Q_1 - (P_1 Q_1 + P_2 Q_2 + P_3 Q_3)R_1 \\ &= (\mathbf{P} \cdot \mathbf{R})Q_1 - (\mathbf{P} \cdot \mathbf{Q})R_1. \end{aligned}$$

Similarly,

$$\begin{aligned} U_2 &= (\mathbf{P} \cdot \mathbf{R})Q_2 - (\mathbf{P} \cdot \mathbf{Q})R_2, \\ U_3 &= (\mathbf{P} \cdot \mathbf{R})Q_3 - (\mathbf{P} \cdot \mathbf{Q})R_3. \end{aligned}$$

These three expressions for  $U_1$ ,  $U_2$ ,  $U_3$  can be combined into the vector equation

$$(9.202) \quad \mathbf{U} = \mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = (\mathbf{P} \cdot \mathbf{R})\mathbf{Q} - (\mathbf{P} \cdot \mathbf{Q})\mathbf{R}.$$

The following remark is an aid in remembering this expression: since  $\mathbf{Q} \times \mathbf{R}$  is perpendicular to the plane of  $\mathbf{Q}$  and  $\mathbf{R}$ , the vector  $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$  must be in this plane; hence,

$$\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = q\mathbf{Q} + r\mathbf{R},$$

where  $q$  and  $r$  are scalars.

*Exercise.* Evaluate all the vector triple products of the unit coordinate vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , including those in which one of the vectors is repeated.

### 9.3. MOMENTS OF VECTORS

The moment of a vector about a line was defined in Sec. 2.3 as a scalar. There we spoke also of “the moment of a vector about a point  $A$ ,” but only as an abbreviation for “the moment about a line through  $A$  perpendicular to the plane containing  $A$  and the vector.” Now that we are in possession of the powerful vector notation, we shall make a fresh start. We shall define the vector moment of a vector about a point, and (in terms of it) the scalar moment of a vector about a line; this latter definition will be shown to agree with that given in Sec. 2.3.

## Moment of a vector about a point.

Let  $\mathbf{P}$  be a vector with origin at  $B$ , and  $A$  any point in space (Fig. 103). We define the *vector moment* of  $\mathbf{P}$  about  $A$  (or briefly the *moment* of  $\mathbf{P}$  about  $A$ ) as a vector  $\mathbf{M}$ , given by

$$(9.301) \quad \mathbf{M} = \mathbf{r} \times \mathbf{P},$$

where  $\mathbf{r} = \overrightarrow{AB}$ , the position vector of  $B$  relative to  $A$ . Thus  $\mathbf{M}$

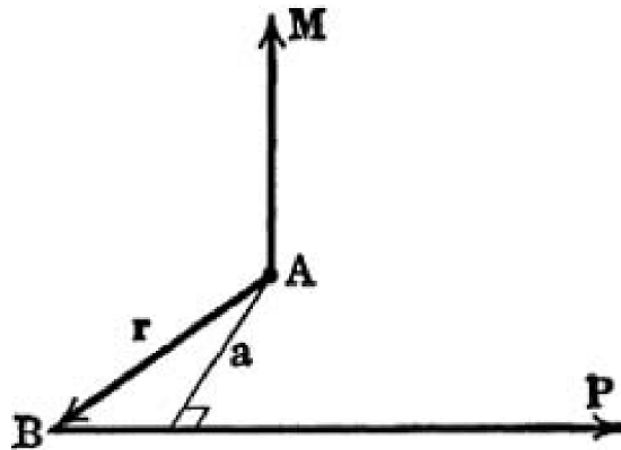


FIG. 103.—The moment of vector about a point.

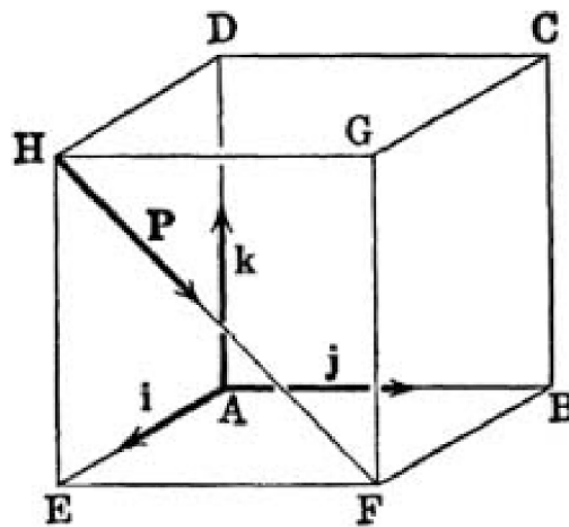


FIG. 104.—The moment of  $\mathbf{P}$  about  $A$  is required.

is a vector perpendicular to the plane of  $\mathbf{r}$  and  $\mathbf{P}$ , with magnitude

$$(9.302) \quad M = rP \sin \theta = aP,$$

where  $\theta$  is the angle between  $\mathbf{r}$  and  $\mathbf{P}$ , and  $a$  the perpendicular from  $A$  on the line of action of  $\mathbf{P}$ .

As an illustration, let us calculate the moment of a given force  $\mathbf{P}$  about a point  $A$ . In



Fig. 104,  $\mathbf{P}$  is a force of known magnitude applied at  $H$  and acting along the diagonal  $HF$  of one face of the cube  $ABCD \dots H$ . If  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  is a triad of unit orthogonal vectors at  $A$  (as shown), we have

$$\begin{aligned}\vec{AH} &= b(\mathbf{i} + \mathbf{k}), \\ \mathbf{P} &= \frac{P}{\sqrt{2}} (\mathbf{j} - \mathbf{k}),\end{aligned}$$

where  $b$  denotes an edge of the cube. The moment  $\mathbf{M}$  of  $\mathbf{P}$  about  $A$  is now easily calculated; it is

$$\mathbf{M} = b(\mathbf{i} + \mathbf{k}) \times \frac{P}{\sqrt{2}} (\mathbf{j} - \mathbf{k}).$$

Hence, by (9.108),

$$\mathbf{M} = \frac{bP}{\sqrt{2}} (-\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Thus the moment of  $\mathbf{P}$  about  $A$  is a vector with components  $(-bP/\sqrt{2}, bP/\sqrt{2}, bP/\sqrt{2})$  in the directions of the edges  $AE, AB, AD$ , respectively.

Returning to the situation shown in Fig. 103, let us investigate the effect of sliding  $\mathbf{P}$  along its line of action. It becomes

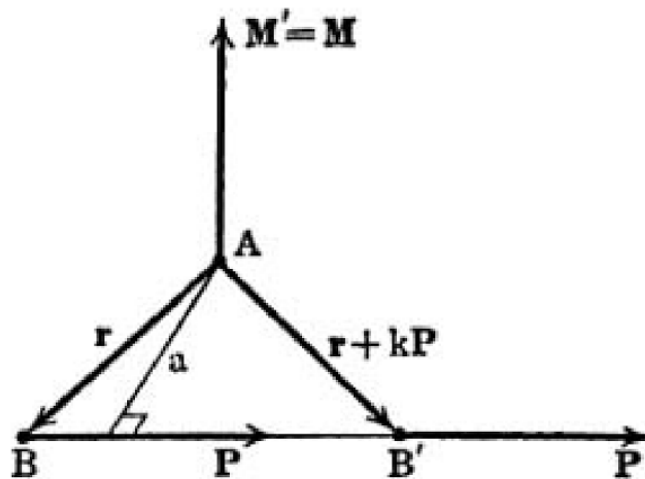


FIG. 105.—The moment of a vector is unchanged when we slide the vector along its line of action.

(Fig. 105) a vector  $\mathbf{P}$  at  $B'$ , where  $\vec{AB'} = \mathbf{r} + k\mathbf{P}$  ( $k$  being some scalar). The moment about  $A$  is now

$$\mathbf{M}' = (\mathbf{r} + k\mathbf{P}) \times \mathbf{P}.$$

But  $\mathbf{P} \times \mathbf{P} = 0$ , and hence

$$\mathbf{M}' = \mathbf{r} \times \mathbf{P} = \mathbf{M}.$$

Thus *the moment of a vector about a point is unaltered by sliding the vector along its line of action.*

We shall now make an important deduction from the above fact. Let  $\mathbf{P}$  at  $B$  and  $-\mathbf{P}$  at  $B'$  be two vectors with a common line of action  $L$ , and let  $A$  be any point. Sliding  $-\mathbf{P}$  along  $L$  until its origin is at  $B$ , we do not alter its moment about  $A$ .

Thus, if  $\overrightarrow{AB} = \mathbf{r}$ , the sum of the moments about  $A$  of  $\mathbf{P}$  at  $B$  and  $-\mathbf{P}$  at  $B'$  is

$$\mathbf{r} \times \mathbf{P} + \mathbf{r} \times (-\mathbf{P}) = \mathbf{r} \times (\mathbf{P} - \mathbf{P}) = 0.$$

In words, for two vectors in the same line, with equal magnitudes but opposite senses, the vector sum of moments about any point is zero. In particular, by the fundamental law of action and reaction (cf. Sec. 1.4), we have

(9.303) *The vector sum of moments about an arbitrary point of the forces of interaction between two particles of any system is zero.*

By the distributive law for vector multiplication, we have, for any vectors,

$$(9.304) \quad \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q} + \mathbf{r} \times \mathbf{R} + \dots = \mathbf{r} \times (\mathbf{P} + \mathbf{Q} + \mathbf{R} + \dots).$$

Let  $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \dots$  be vectors with common origin  $B$ , and let  $\mathbf{r}$  be the position vector of  $B$  relative to a point  $A$ . Then, for vector moments about a point, we have the theorem of Varignon (cf. Sec. 2.3): *The sum of the vector moments about a point  $A$  of vectors  $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \dots$  with common origin  $B$ , is equal to the vector moment about  $A$  of the single vector  $\mathbf{P} + \mathbf{Q} + \mathbf{R} + \dots$  with origin  $B$ .*

## Moment of a vector about a line.

Let  $\mathbf{M}$  be the moment of a vector  $\mathbf{P}$  about a point  $A$ , and let  $L$  be any line through  $A$ . Of the two senses on  $L$ , we choose one as positive and distinguish it by a unit vector  $\lambda$  lying on  $L$ . We define *the scalar moment of  $\mathbf{P}$  about  $L$*  as the component  $M_\lambda$  of  $\mathbf{M}$  in the positive sense of  $L$ ; expressed in symbols,

$$(9.305) \quad M_\lambda = \lambda \cdot \mathbf{M}.$$

We shall now show that the above definition is equivalent to that given in Sec. 2.3. We take special axes  $Oxyz$  as shown in Fig. 106; the origin  $O$  coincides with  $A$ , and  $Oz$  lies along the line  $L$  in the positive sense. Relative to these axes,  $\mathbf{P}$  has components  $(X, Y, Z)$  and acts at a point  $B$  with coordinates  $(x, y, z)$ . The moment  $\mathbf{M}$  of  $\mathbf{P}$  about  $A$  (or  $O$ ) is

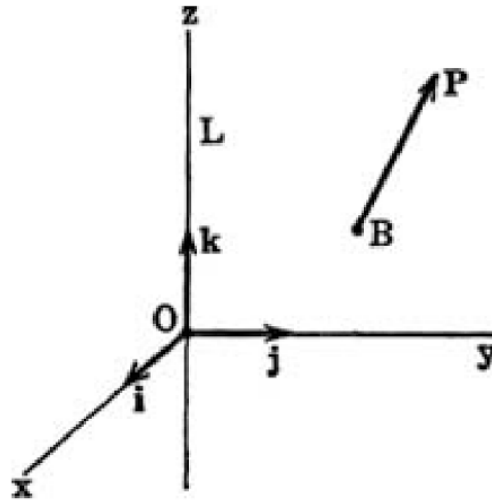


FIG. 106.—The moment of  $\mathbf{P}$  about the line  $L$  is required.

$$(9.306) \quad \begin{aligned} \mathbf{M} &= (xi + yj + zk) \times (Xi + Yj + Zk) \\ &= (yZ - zY)i + (zX - xZ)j + (xY - yX)k, \end{aligned}$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the unit coordinate vectors. Since  $\lambda = \mathbf{k}$ , we have

$$(9.307) \quad M_\lambda = \mathbf{k} \cdot \mathbf{M} = xY - yX.$$

But this quantity is precisely the moment as given by (2.303); the two definitions of the moment of a vector about a line are now completely reconciled.

It might appear that the value of the moment  $M_\lambda$  of  $\mathbf{P}$  about  $L$ , as given by (9.305), depends on the choice of a point  $A$  on this line. This is not actually the case. For let  $A'$  be any other

point on  $L$ , so that  $\vec{AA'} = k\lambda$ , where  $k$  is some scalar. The moment about  $A'$  of  $\mathbf{P}$  at  $B$  is

$$\mathbf{M}' = (-k\lambda + \mathbf{r}) \times \mathbf{P},$$

where  $\mathbf{r} = \vec{AB}$ . The component of  $\mathbf{M}'$  in the positive sense of  $L$  is therefore

$$\begin{aligned}
M_{\lambda} &= \lambda \cdot [(-k\lambda + \mathbf{r}) \times \mathbf{P}] \\
&= -k\lambda \cdot (\lambda \times \mathbf{P}) + \lambda \cdot (\mathbf{r} \times \mathbf{P}) \\
&= \lambda \cdot \mathbf{M} = M_{\lambda},
\end{aligned}$$

since  $\lambda \cdot (\lambda \times \mathbf{P}) = 0$ .

The theorem of Varignon for scalar moments of vectors about a line follows directly from (9.304); we have merely to take the scalar product of each side with  $\lambda$ . This very simple proof by vector methods should be compared with that of Sec. 2.3, where only elementary methods were used.

There are occasions, however, where scalar methods are more direct than vector methods. On such occasions, we require formulas for the moments of a vector about the axes of coordinates. These are, by (9.306),

$$(9.308) \quad yZ - zY, \quad zX - xZ, \quad xY - yX,$$

where  $(X, Y, Z)$  are the components of the vector applied at  $(x, y, z)$ .

*Exercise.* A vector with components  $(1, 2, 3)$  acts at the point  $(3, 2, 1)$ . What is its moment about the origin, and what are its moments about the coordinate axes?

## 9.4. SUMMARY OF PRODUCTS OF VECTORS

### I. Scalar product.

$$(9.401) \quad \mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} = PQ \cos \theta = P_1Q_1 + P_2Q_2 + P_3Q_3.$$

### II. Vector product.

$$(9.402) \quad \mathbf{P} \times \mathbf{Q} = -\mathbf{Q} \times \mathbf{P} = PQ \sin \theta \mathbf{n}.$$

( $\mathbf{n}$  a unit vector perpendicular to  $\mathbf{P}$  and  $\mathbf{Q}$ ; triad  $\mathbf{P}, \mathbf{Q}, \mathbf{n}$  right-handed.)

**III. Usual rules of algebra and calculus apply to products of vectors, if order in vector products is preserved.**

### IV. The mixed triple product.

$$(9.403) \quad \mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix}.$$

### V. The vector triple product.

$$(9.404) \quad \mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = (\mathbf{P} \cdot \mathbf{R})\mathbf{Q} - (\mathbf{P} \cdot \mathbf{Q})\mathbf{R}.$$

## VI. Moment of a vector about a point.

$$(9.405) \quad \mathbf{M} = \mathbf{r} \times \mathbf{P} \\ = (yZ - zY)\mathbf{i} + (zX - xZ)\mathbf{j} + (xY - yX)\mathbf{k}.$$

## VII. Moment of a vector about a directed line ( $\lambda$ ).

$$(9.406) \quad M_\lambda = \lambda \cdot (\mathbf{r} \times \mathbf{P}).$$

### EXERCISES IX

1. Solve the equations

$$2\mathbf{A} + \mathbf{B} = \mathbf{M}, \mathbf{A} + 2\mathbf{B} = \mathbf{N},$$

$\mathbf{M}$  and  $\mathbf{N}$  being given vectors.

2. Three vectors are represented by the diagonals of three adjacent faces of a cube, all passing through the same corner and directed away from it. Find their sum.

3. What is the moment about the  $x$ -axis of a force of magnitude 3 applied at a point with coordinates  $(2, 3, 5)$ , in a direction making angles of  $60^\circ$  with the axes of  $y$  and  $z$  and an acute angle with the axis of  $x$ ?

4.  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  are any four vectors. Prove that there exist scalars  $a, b, c, d$  (not all zero), such that

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D} = \mathbf{0}.$$

5. If  $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ , show that  $\mathbf{B} = \mathbf{C} + k\mathbf{A}$ , where  $k$  is some scalar.

6. If  $\mathbf{A}$  and  $\mathbf{B}$  are any two unit vectors, prove that the moment of  $\mathbf{A}$  about  $\mathbf{B}$  is equal to the moment of  $\mathbf{B}$  about  $\mathbf{A}$ .

7. Find the moments, about a corner of a cube, of three unit vectors converging on the opposite corner along three edges. Show that the sum of the moments is zero. How could you obtain this result without calculation?

8. A force with components  $(X, Y, Z)$  acts at the point  $(a, b, c)$ . What is its moment about a line through the origin with direction cosines  $(l, m, n)$ ?

9. A force of magnitude  $P$  acts along the line joining opposite corners of a cube of edge  $2a$ . Find the moment of the force about a line which is a diagonal of a face of the cube and which does not cut the line of action of the force.

10. A directed line  $L$  passes through the point  $(a, b, c)$  with direction cosines  $(l, m, n)$ . Prove that the moment about  $L$  of a unit vector pointing along the  $x$ -axis is  $bn - cm$ .

11. Prove that the moment of a vector about a line vanishes if, and only if, the

vector cuts the line or is parallel to it.

12. Prove the identities

- (i)  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0},$
- (ii)  $\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D}),$
- (iii)  $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{B}[\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - \mathbf{A}[\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D})].$

13.  $Oxyz$ ,  $Ox'y'z'$  are two sets of rectangular Cartesian axes.  $\mathbf{P}$  is a vector with components  $X, Y, Z$  on  $Oxyz$  and components  $X', Y', Z'$  on  $Ox'y'z'$ . Show that

$$X' = a_{11}X + a_{12}Y + a_{13}Z,$$

where  $a_{11}, a_{12}, a_{13}$  are the direction cosines of  $Ox'$  with respect to  $Oxyz$ . Develop similar formulas for  $Y'$  and  $Z'$ .

14. Solve the differential equation

$$\frac{d^2 \mathbf{r}}{dt^2} = \mathbf{a},$$

where  $\mathbf{a}$  is a constant vector.

15. Show that the differential equation

$$\mathbf{a} \times \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{b},$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular constant vectors, has the general solution

$$\mathbf{r} = f(t)\mathbf{a} + t\mathbf{c} + \mathbf{e} - \frac{t^2}{2a^2}(\mathbf{a} \times \mathbf{b});$$

here  $f(t)$  is an arbitrary function and  $\mathbf{c}, \mathbf{e}$  are arbitrary constant vectors.

is called a *geodesic* on  $S$ .

## Flexible cables.

Let us now consider a flexible cable in equilibrium under the action of known external forces and the tensions at its ends.

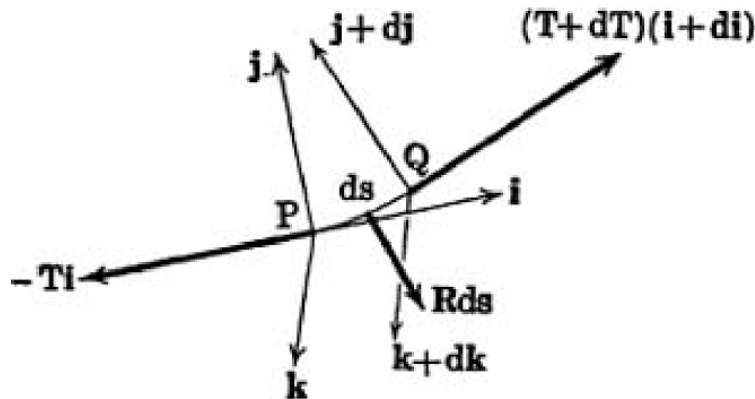


FIG. 107.—Forces on an element of cable.

Figure 107 shows an infinitesimal portion  $PQ$ ,  $P$  being at a distance  $s$  from one end of the cable. Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  denote the unit tangent, principal normal, and binormal vectors at  $P$ . The forces acting on the element  $PQ$  (length  $ds$ ) may now be described as follows:

- (i) a force  $-T\mathbf{i}$  at  $P$ , where  $T$  is the tension at  $P$ ;
- (ii) a force  $(T + dT)(\mathbf{i} + d\mathbf{i}) = (T + dT)[\mathbf{i} + (\mathbf{j}/\rho) ds]$  at  $Q$ , where  $T + dT$  is the tension at  $Q$ ;
- (iii) a force  $\mathbf{R} ds = (R_1\mathbf{i} + R_2\mathbf{j} + R_3\mathbf{k}) ds$ , where  $\mathbf{R}$  is the external force per unit length of the cable. (This force acts at some unspecified point of the element  $PQ$ .)

The element  $PQ$  is a system in equilibrium under these forces, and so we may apply the conditions (10.207) to it. From the first of these conditions, we have

$$(10.216) \quad -T\mathbf{i} + (T + dT) \left( \mathbf{i} + \frac{\mathbf{j}}{\rho} ds \right) + (R_1\mathbf{i} + R_2\mathbf{j} + R_3\mathbf{k}) ds = \mathbf{0}.$$

The second of the conditions (10.207) is satisfied identically to the first order in  $ds$ . From (10.216), we at once obtain the scalar equations



space. The vector

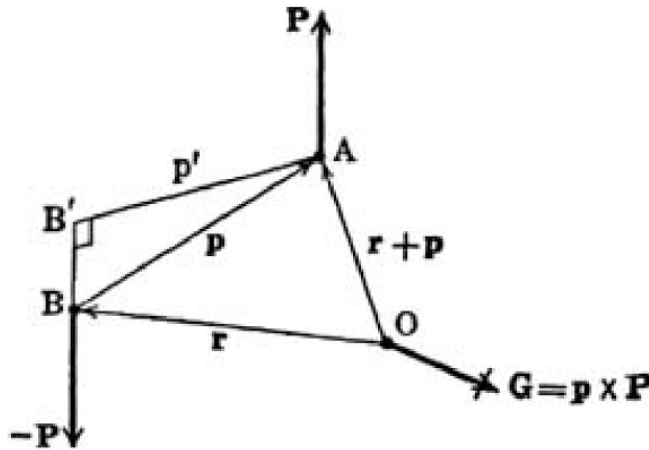


FIG. 108.—The moment of a couple.

moment  $\mathbf{G}$  of this couple about  $O$  is easily found; denoting  $\vec{OB}$  by  $\mathbf{r}$  and  $\vec{BA}$  by  $\mathbf{p}$ , we obtain

$$(10.301) \quad \mathbf{G} = \mathbf{r} \times (-\mathbf{P}) + (\mathbf{r} + \mathbf{p}) \times \mathbf{P} = \mathbf{p} \times \mathbf{P}.$$

This value of  $\mathbf{G}$  is independent of the position of the point  $O$ . In other words, *a couple has the same moment about all points in space*. Thus the vector  $\mathbf{G}$  may be regarded as a free vector; it is perpendicular to the plane determined by the forces  $\mathbf{P}$ ,  $-\mathbf{P}$  of the couple; its magnitude is  $p'P$ , where  $p'$  is the perpendicular distance between the lines of action of these forces (Fig. 108).

Since two couples which have the same moment are equipollent force systems, a couple is completely specified (as far as equipollence is concerned) by its free moment vector, or, briefly, its *moment*. Thus, when we speak of a couple  $\mathbf{G}$ , we have in mind any one of an infinite number of couples, each of which has moment  $\mathbf{G}$ .

To avoid confusion in diagrams, the arrowheads indicating couples may be marked with a crossbar, as in the figure.

### Composition of couples.

Let there be a system of forces consisting of a number of couples  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ , . . . . The total force of this system is zero, and the total moment about any point is clearly

$$(10.302) \quad \mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2 + \cdots$$

Thus, *a system consisting of couples is equipollent to a single couple; its moment is equal to the vector sum of the moments of the individual couples*. In

other words, couples are compounded by the parallelogram law.

*Exercise.* Forces with components  $(2, 0, 0)$ ,  $(-1, 0, 0)$ ,  $(-1, 0, 0)$  act at the points  $(0, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ , respectively. Show that they can be reduced to a couple, and find its magnitude and direction.

## Reduction of a force system to a force and a couple.

Consider a general force system  $S$ , with total force  $\mathbf{F}$  and total moment  $\mathbf{G}$  with respect to a base point  $O$ . Consider also a second force system  $S'$ , consisting only of a single force  $\mathbf{F}$  applied at  $O$  and a single couple  $\mathbf{G}$ . Obviously,  $S'$  is equipollent to  $S$ . *Therefore a general force system can always be reduced to a single force applied at an arbitrary base point, together with a couple.*

Just as we represent a single force by an arrow, so we can represent a general force system by a diagram such as that in Fig. 109; this shows a force  $\mathbf{F}$  acting at a base point  $O$  and a couple  $\mathbf{G}$ . Although the couple  $\mathbf{G}$  is a free vector, it is convenient to draw it out from the base point  $O$ .

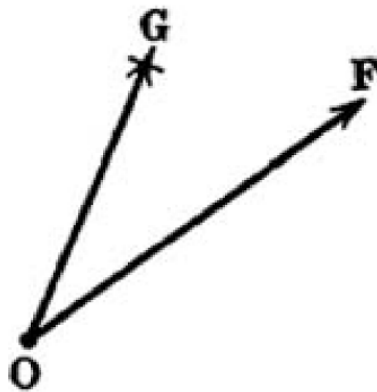


FIG. 109.—Representation of a general force system by a force  $\mathbf{F}$  and a couple  $\mathbf{G}$ .

If we change the base point from  $O$  to  $O'$ , where  $\overrightarrow{OO'} = \mathbf{r}$ , we do not alter  $\mathbf{F}$ , but the moment about  $O'$  is not  $\mathbf{G}$ ; it is found by adding to  $\mathbf{G}$  the moment of  $\mathbf{F}$  about  $O'$ ; this gives [cf. (10.106)]

$$\mathbf{G}' = \mathbf{G} - \mathbf{r} \times \mathbf{F}.$$

Hence, under a change of base point from  $O$  to  $O'$ , the force  $\mathbf{F}$  and the couple  $\mathbf{G}$  become  $\mathbf{F}'$  and  $\mathbf{G}'$ , respectively, where

$$(10.303) \quad \mathbf{F}' = \mathbf{F}, \quad \mathbf{G}' = \mathbf{G} - \mathbf{r} \times \mathbf{F}.$$

We note that  $F' = F$  and  $\mathbf{F}' \cdot \mathbf{G}' = \mathbf{F} \cdot \mathbf{G}$ ; in words, the scalars  $F$  and  $\mathbf{F} \cdot \mathbf{G}$  are *invariant* under a change of base point. If either of these invariants vanishes for one choice of base point, then it vanishes for all choices of base point.

## Reduction to a wrench.

A *wrench* consists of a force  $\mathbf{F}$  and a couple  $\mathbf{G}$  with parallel representative line segments. This relation between  $\mathbf{F}$  and  $\mathbf{G}$  is expressed by the vector equation

$$(10.304) \quad \mathbf{G} = p\mathbf{F},$$

where  $p$  is some scalar having the dimensions of a length. The quantities  $p$  and  $F$  are called the *pitch* and *intensity* of the wrench, respectively. The line of action of the force  $\mathbf{F}$  is called the *axis* of the wrench.

A general force system can always be reduced to a wrench. We shall now show how this is done. Let us first reduce the system in question to a force  $\mathbf{F}$  at a base point  $O$  and a couple  $\mathbf{G}$ .

Changing the base point to  $O'$ , where  $\vec{OO'} = \mathbf{r}$ , we obtain a force  $\mathbf{F'}$  and couple  $\mathbf{G'}$ . These constitute a wrench if

$$(10.305) \quad \mathbf{G'} = p\mathbf{F'}.$$

Since, by (10.303),

$$\mathbf{F'} = \mathbf{F}, \quad \mathbf{G'} = \mathbf{G} - \mathbf{r} \times \mathbf{F},$$

the equation (10.305) is satisfied if  $\mathbf{r}$  and  $p$  satisfy

$$(10.306) \quad \mathbf{G} - \mathbf{r} \times \mathbf{F} = p\mathbf{F}.$$

This is, in fact, a vector equation for  $\mathbf{r}$  (the position vector of  $O'$ ) and  $p$  (the pitch of the wrench).

Let us take  $O$  as origin of rectangular Cartesian coordinates, and let  $(F_1, F_2, F_3)$ ,  $(G_1, G_2, G_3)$  denote the components of  $\mathbf{F}$ ,  $\mathbf{G}$ , respectively. The vector equation (10.306) is equivalent to the scalar equations

$$\frac{G_1 - yF_3 + zF_2}{F_1} = \frac{G_2 - zF_1 + xF_3}{F_2} = \frac{G_3 - xF_2 + yF_1}{F_3} = p,$$

where  $x, y, z$  are the coordinates of  $O'$ . These equations show that  $\mathbf{F'}$ ,  $\mathbf{G'}$  constitute a wrench provided  $O'$  lies on the straight line with equations

$$(10.307) \quad \frac{G_1 - yF_3 + zF_2}{F_1} = \frac{G_2 - zF_1 + xF_3}{F_2} = \frac{G_3 - xF_2 + yF_1}{F_3}.$$

We note that, if  $(x, y, z)$  is any point on this line, then  $(x + kF_1, y + kF_2, z + kF_3)$ , where  $k$  is any scalar factor, is also on it. It follows that this line has the

direction of  $\mathbf{F}$ ; it is the axis of the wrench to which the system is reduced.

The pitch  $p$  is found by taking the scalar product of  $\mathbf{F}$  and the vectors on the two sides of (10.306). We find

$$\mathbf{F} \cdot \mathbf{G} = pF^2;$$

therefore,

$$(10.308) \quad p = \frac{\mathbf{F} \cdot \mathbf{G}}{F^2}.$$

It is, of course, not accidental that the pitch of the resulting wrench is a function of the invariants  $F$  and  $\mathbf{F} \cdot \mathbf{G}$ .

If  $p = 0$ , the wrench degenerates into a single force; in this case  $\mathbf{F} \cdot \mathbf{G} = 0$ . If  $p$  is infinite, the wrench degenerates into a couple; in this case  $F = 0$ . In each of these special cases, we have a force system equipollent to a plane system of forces. Conversely, if the force system is equipollent to a plane system of forces, then one or other of these special cases must arise.

*Exercise.* In the reduction of a given force system to a force and a couple, the couple  $\mathbf{G}$  depends on the base point. For what base points is  $G$  least?

### Reduction of a system of parallel forces.

A set of parallel forces is a system of particular importance in mechanics, e.g., the weights of a number of particles.

Any  $n$  parallel forces may be denoted by  $k_1\mathbf{P}, k_2\mathbf{P}, \dots, k_n\mathbf{P}$ , where  $k_1, k_2, \dots, k_n$  are scalars. Selecting a base point  $O$ , we first reduce this system to a force  $\mathbf{F}$  at  $O$  and a couple  $\mathbf{G}$ . Let  $\mathbf{r}_s$  ( $s = 1, 2, \dots, n$ ) denote the position vectors, relative to  $O$ , of the points of application of the several forces. Then,

$$(10.309) \quad \begin{cases} \mathbf{F} = \sum_{s=1}^n (k_s\mathbf{P}) = k\mathbf{P}, \\ \mathbf{G} = \sum_{s=1}^n (\mathbf{r}_s \times k_s\mathbf{P}) = \left( \sum_{s=1}^n k_s\mathbf{r}_s \right) \times \mathbf{P} = \mathbf{r} \times \mathbf{F}, \end{cases}$$

where

$$(10.310) \quad k = \sum_{s=1}^n k_s, \quad \mathbf{r} = \left( \sum_{s=1}^n k_s\mathbf{r}_s \right) / k.$$

This reduced system is clearly equipollent to the single force  $\mathbf{F}$ , applied at the point  $C$  with position vector  $\mathbf{r}$ .

Thus a system of parallel forces can be reduced to a single force, unless  $k = 0$ . If  $k = 0$ , it can be reduced to a couple.

The point  $C$ , with position vector  $\mathbf{r}$  given by (10.310), is called the *center of the system of parallel forces*. Its position is determined solely by the vectors  $\mathbf{r}_s$  and the ratios of the numbers  $k_1, k_2, \dots, k_n$ . It is unaltered by turning the several forces about their points of application, provided they retain their magnitudes and remain parallel to one another.

If the forces in question are the weights of the particles of a system, the point  $C$  is the center of gravity of the system (cf. Sec. 3.1).

## The reduction of some special force systems.

The force systems encountered in practical problems are often extremely complicated. However, the details of such a force system are relatively unimportant when it acts on a rigid body; if we know the *total force* and the *total moment*, we know all that is essential for the discussion of equilibrium. The total force and the total moment play an equally important part in dynamics (see Chap. XII).

### 1. Analysis of forces on an airplane.

Figure 110 shows an airplane in flight.  $C$  is the mass center. The orthogonal right-handed triad of unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  is fixed in the airplane;  $\mathbf{j}$  is perpendicular to the plane of symmetry and points to the right;  $\mathbf{i}$  and  $\mathbf{k}$  lie in the plane of symmetry. The direction of  $\mathbf{i}$  is fixed in some conventional way (e.g., parallel to the airscrew axes), so as to be nearly horizontal and point forward when the plane is in normal flight;  $\mathbf{k}$  will then be directed nearly vertically downward.

The forces acting on the airplane are as follows:

(i) The weights of the various parts. These constitute a system of parallel forces and can be reduced to a single force  $\mathbf{W}$  acting vertically downward through  $C$ ;  $W$  is the total weight of the airplane.

(ii) The thrust, or driving force,  $\mathbf{P}$  due to the airscrews. This force is in the direction of the vector  $\mathbf{i}$  or nearly so.

(iii) Forces arising from the action of the air. These forces are due mainly to variations in pressure over the wing surface and in a lesser degree to friction; they form a very complicated system. Reducing this system to a force  $\mathbf{F}$  at  $C$  and a couple  $\mathbf{G}$ , we resolve as follows:

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}, \quad \mathbf{G} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}.$$



For normal flight,  $-X$  is the *drag*,  $-Z$  is the *lift* and  $M$  is the *pitching moment*;  $Y$ ,  $L$ ,  $N$  are zero. The precise terminology of aerodynamic theory is not quite so simple as this. Under normal flight conditions, however, the differences are small.

The theoretical determination of the force system  $(\mathbf{F}, \mathbf{G})$  is a very difficult problem in hydrodynamics; and, in practice, experimental methods are used. A model of the airplane (or the airplane itself) is mounted in a wind tunnel. Direct measurements are then made of the force system required to keep the model at rest in a stream of air.

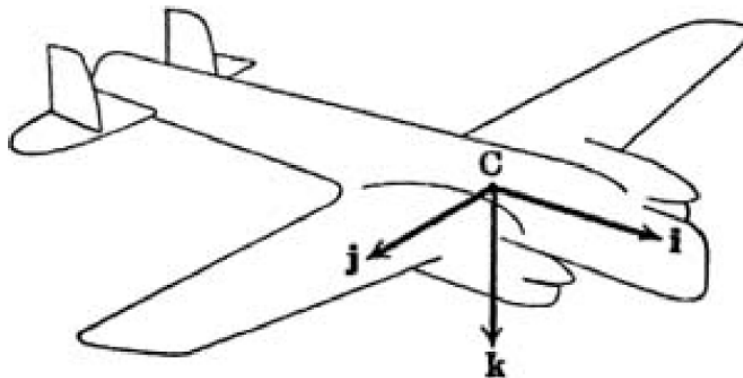


FIG. 110.—Reference vectors for an airplane.

An analysis of the forces on a bullet or shell follows the same lines. In this case, however, the driving force  $\mathbf{P}$  is absent.

## 2. Analysis of stresses in a beam.

Consider a beam in equilibrium and let  $Ox$  be a line in the direction of its length. We imagine the beam cut in two by a plane  $\Pi$  perpendicular to  $Ox$  at  $A$  (Fig. 111). We shall denote by  $R$  the part of the beam to the right of  $\Pi$ , and by  $L$  the part to the left.

The forces on  $L$  are as follows:

(i) Applied forces, such as gravity or external loads. These are equipollent to a single force  $\mathbf{F}$  at  $A$  and a couple  $\mathbf{G}$ .

(ii) Forces exerted across  $\Pi$  by  $R$  on  $L$ . These are internal forces for the whole beam, but external forces for the system  $L$ . They are called the *stresses* across the plane section  $\Pi$  and are equipollent to a force  $\mathbf{S}$  at  $A$  and a couple  $\mathbf{M}$ . Introducing the orthogonal triad of unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , as shown, we write

$$\mathbf{S} = S_1\mathbf{i} + S_2\mathbf{j} + S_3\mathbf{k}, \quad \mathbf{M} = M_1\mathbf{i} + M_2\mathbf{j} + M_3\mathbf{k}.$$

The following terminology is used:

$S_1 = \text{tension,}$   
 $S_2, S_3 = \text{shearing forces,}$   
 $M_1 = \text{twisting couple,}$   
 $M_2, M_3 = \text{bending moments.}$

If the applied forces are known, then  $\mathbf{F}$ ,  $\mathbf{G}$  are known and we can find  $\mathbf{S}$ ,  $\mathbf{M}$ . We have merely to apply the conditions of equilibrium (10.207) to  $L$ , obtaining

$$\mathbf{S} = -\mathbf{F}, \quad \mathbf{M} = -\mathbf{G}.$$

If we change the section II by varying the distance  $x$  of  $A$  from  $O$ ,  $\mathbf{F}$  and  $\mathbf{G}$  are known functions of  $x$ ; hence,  $\mathbf{S}$  and  $\mathbf{M}$  are known functions of  $x$ . In other words, there are two vector functions  $\mathbf{S}(x)$ ,  $\mathbf{M}(x)$  [or six scalar functions  $S_1(x)$ ,  $S_2(x)$ , . . .  $M_3(x)$ ] which give, for each value of  $x$ , a force system equipollent to the stresses across the corresponding cross section of the beam.

Failure in an engineering structure, such as a bridge, is due to excessive stress. The engineer must know in advance if any given beam or girder is likely to fail under the loads which it will be called on to support.

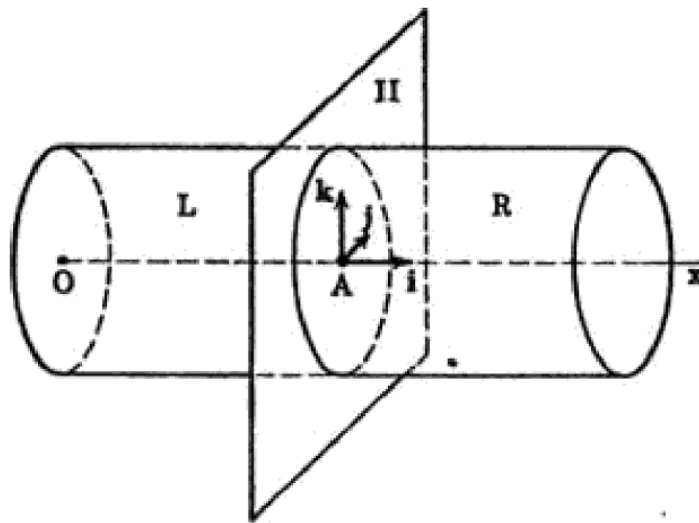


FIG. 111.—Reference vectors for reactions in a beam.

Although the values of  $\mathbf{S}$  and  $\mathbf{M}$  do not give a complete picture of the internal stresses, they are the quantities which the engineer calculates in order to see whether or not a structure is safe.

The above analysis also applies in naval architecture. Regarding the hull of a ship as a beam, subject to known applied forces, we can determine a force system  $(\mathbf{S}, \mathbf{M})$  equipollent to the internal stresses across any section perpendicular to its length. A ship must be so constructed that it will withstand



the action of stresses (**S**, **M**) arising from the applied forces of weight and buoyancy. In a storm the ship may be supported by waves under bow and stern; then the forces of buoyancy are concentrated there, and the ship is in danger of “breaking its back.”

#### 10.4. EQUILIBRIUM OF A RIGID BODY

##### Necessary and sufficient conditions of equilibrium.

In our mathematical model, a rigid body is a set of particles whose mutual distances are invariable. Let us now consider a rigid body acted on by external forces. Reducing the force system to a single force **F** at a base point *O* and a couple **G**, we know by (10.207) that the conditions

$$(10.401) \quad \mathbf{F} = \mathbf{0}, \quad \mathbf{G} = \mathbf{0}$$

are *necessary* for equilibrium. We now make use of the rigidity of the body to prove that these conditions are also *sufficient*, so that the body must be in equilibrium if they are satisfied.

Let us suppose that a rigid body acted on by external forces, satisfying (10.401), is not in equilibrium; then the particles of the body will be on the point of moving. This motion will be prevented by introducing the following constraints (Fig. 112):

(i) The point *O* (taken in the body) is fixed; this leaves the body free to turn about *O*.

(ii) With origin *O*, we draw a unit vector **i**; the particle *A* at its extremity is constrained to slide in a smooth tube with axis in the direction of **i**. The two constraints now introduced fix all points of the body on the line *OA*, but still permit the body to turn about this line.

(iii) With origin *O*, we draw a unit vector **j**, perpendicular to **i**; the particle *B* at its extremity is constrained to move between two smooth planes parallel to the plane *OAB*. If these planes are close to each other, this constraint will prevent the motion of *B*.

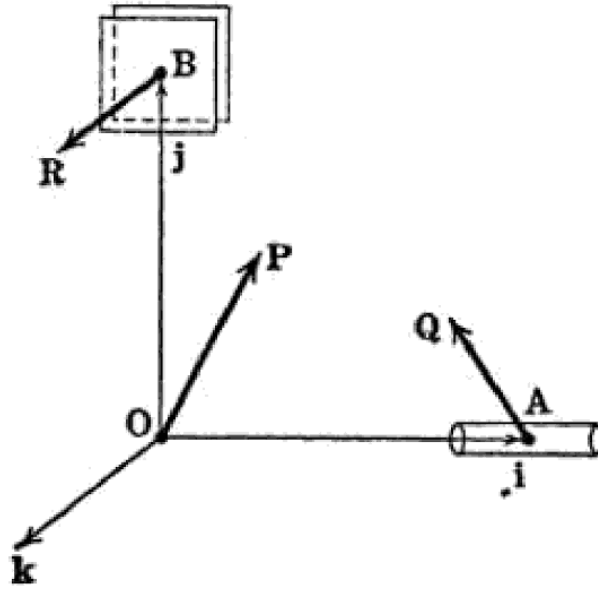


FIG. 112.—Constraints preventing motion of a rigid body.

These three constraints together prevent any motion of the body, and so it must remain at rest. It is therefore in equilibrium under the action of the given external force system and the reactions of these constraints.

Now the reactions of constraint are equipollent to a force  $\mathbf{F}'$  at  $O$  and a couple  $\mathbf{G}'$ . Since the body is in equilibrium,

$$\mathbf{F} + \mathbf{F}' = \mathbf{0}, \quad \mathbf{G} + \mathbf{G}' = \mathbf{0};$$

and so, by (10.401),

$$(10.402) \quad \mathbf{F}' = \mathbf{0}, \quad \mathbf{G}' = \mathbf{0}.$$

In view of the smoothness of the constraints at  $A$  and  $B$ , we see that the forces of constraint are

- (i) a force  $\mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k}$  applied at  $O$ ;
- (ii) a force  $\mathbf{Q} = Q_2\mathbf{j} + Q_3\mathbf{k}$  applied at  $A$  (position vector  $\mathbf{i}$  relative to  $O$ );
- (iii) a force  $\mathbf{R} = R_3\mathbf{k}$  applied at  $B$  (position vector  $\mathbf{j}$  relative to  $O$ ).

The vector  $\mathbf{k}$  is a unit vector completing the orthogonal triad  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . On calculating  $\mathbf{F}', \mathbf{G}'$  in terms of  $P_1, P_2, P_3, Q_2, Q_3, R_3$ , we find that the six scalar equations contained in (10.402) imply

$$P_1 = P_2 = P_3 = Q_2 = Q_3 = R_3 = 0.$$

Hence the constraints introduced actually exert no reactions, and so the body remains in equilibrium even if they are removed. The sufficiency of the conditions (10.401) is now established.

It is possible to state the conditions of equilibrium in forms other than (10.401). For example, it is easy to see that, if the external forces have no moment about each of three non-collinear points, then the conditions (10.401) are satisfied. Conversely, if the external forces satisfy (10.401) for some particular base point, they have no moment about any point. Thus, if  $\mathbf{G}, \mathbf{G}', \mathbf{G}''$  denote the total moments of the external force system about each of three non-collinear points, the conditions

$$(10.403) \quad \mathbf{G} = \mathbf{G}' = \mathbf{G}'' = \mathbf{0}$$

are both necessary and sufficient for equilibrium. Occasionally the conditions (10.403) are easier to apply than (10.401).

## Applications.

The conditions (10.401) will now be applied to solve two problems.

*Example 1.* Figure 113 shows a pulley, with radius  $r$  and center  $A$ , rigidly attached to a horizontal shaft  $BCD$ . This shaft is free to turn in smooth bearings at  $B$  and  $C$ ; the end  $D$  projects beyond the bearing at  $C$ , and to it is rigidly fastened a crank  $DE$  with handle  $EH$ . The angles  $BDE$  and  $DEH$  are right angles. A weight  $W$  is attached to the lower end of a cord passing round the pulley, the other end of the cord being fixed to the pulley. To raise  $W$ , a man applies a force  $P$  at  $H$  in a direction perpendicular to  $BCD$  and making an angle  $\phi$  with the horizontal.

It is required to find the magnitude of  $\mathbf{P}$  and also the directions and magnitudes of the reactions  $\mathbf{R}$  and  $\mathbf{R}'$  at  $B$  and  $C$ , respectively.

Let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be an orthogonal triad of unit vectors at  $A$ ,  $\mathbf{i}$  lying along  $AC$  and  $\mathbf{k}$  pointing vertically upward. Lengths are denoted as follows:  $BA = AC = a$ ,  $CD = b$ ,  $DE = c$ ,  $EH = d$ . Then, if  $DE$  makes an angle  $\theta$  with the vertical, the forces acting on the whole system can be described as follows (position vectors being taken relative to  $A$ ):

$$(10.404) \quad \left\{ \begin{array}{l} \text{a force } \mathbf{P} = P \cos \phi \mathbf{j} - P \sin \phi \mathbf{k} \\ \quad \quad \quad \text{at } (a + b + d)\mathbf{i} + c \sin \theta \mathbf{j} + c \cos \theta \mathbf{k}, \\ \text{a force } -W\mathbf{k} \text{ at } -r\mathbf{j}, \\ \text{a force } \mathbf{R} = R_2\mathbf{j} + R_3\mathbf{k} \text{ at } -a\mathbf{i}, \\ \text{a force } \mathbf{R}' = R'_2\mathbf{j} + R'_3\mathbf{k} \text{ at } a\mathbf{i}. \end{array} \right.$$

not only  $ABCD$ , but  $A'B'C'D'$  also. The system has one degree of freedom, and  $\theta$  is a generalized coordinate.

When  $\theta$  increases to  $\theta + \delta\theta$ ,  $A'B'C'D'$  turns through some small angle  $\delta\theta'$ ; the displacements of  $ABCD$ ,  $A'B'C'D'$ ,  $CDC'D'$  are as follows:

$ABCD$ : a rotation  $\delta\theta \mathbf{I}$ ,

$A'B'C'D'$ : a rotation  $\delta\theta' \mathbf{I}'$ ,

$CDC'D'$ : a rotation  $\delta\mathbf{n} = \delta n_1 \mathbf{i} + \delta n_2 \mathbf{j} + \delta n_3 \mathbf{k}$ ,

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are unit vectors along  $CD$ ,  $C'D'$  and perpendicular to them. Since no work is done by the reactions at the bearings and the internal reactions at  $C$ ,  $C'$ ,  $D$ ,  $D'$  we have [cf. (10.704)]

$$(10.715) \quad \begin{aligned} \delta W &= M \mathbf{I} \cdot \delta\theta \mathbf{I} + M' \mathbf{I}' \cdot \delta\theta' \mathbf{I}' \\ &= M \delta\theta + M' \delta\theta'. \end{aligned}$$

We must now find  $\delta\theta'$  in terms of  $\delta\theta$ . The point  $D$  belongs to two rigid bodies  $ABCD$  and  $CDC'D'$ . Equating the two expressions for its displacement [cf (10.501)], we have

$$(10.716) \quad \delta\theta \mathbf{I} \times a \mathbf{i} = \delta\mathbf{n} \times a \mathbf{i},$$

where  $a = OD = OD'$ . Similarly, by considering the displacement of  $D'$ , we find

$$(10.717) \quad \delta\theta' \mathbf{I}' \times a \mathbf{j} = \delta\mathbf{n} \times a \mathbf{j}.$$

Now, resolving along  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , we have

$$\mathbf{I} = -\sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad \mathbf{I}' = -\sin \phi' \mathbf{i} + \cos \phi' \mathbf{k},$$

where  $\phi, \phi'$  are the angles between  $\mathbf{k}$  and  $AB, B'A'$ , respectively. Substituting these values for  $\mathbf{I}$  and  $\mathbf{I}'$  in (10.716) and (10.717), we obtain

$$(10.718) \quad \begin{cases} \cos \phi \delta\theta = \delta n_3, & -\sin \phi \delta\theta = \delta n_2, \\ \cos \phi' \delta\theta' = \delta n_3, & -\sin \phi' \delta\theta' = \delta n_1. \end{cases}$$

Equating the expressions for  $\delta n_3$ , we obtain

$$(10.719) \quad \delta\theta' = \cos \phi \sec \phi' \delta\theta.$$

Substituting this value in (10.715), we get

$$(10.720) \quad \delta W = (M + M' \cos \phi \sec \phi') \delta\theta,$$

and so the single generalized force  $\Theta$  is

$$\Theta = M + M' \cos \phi \sec \phi'.$$

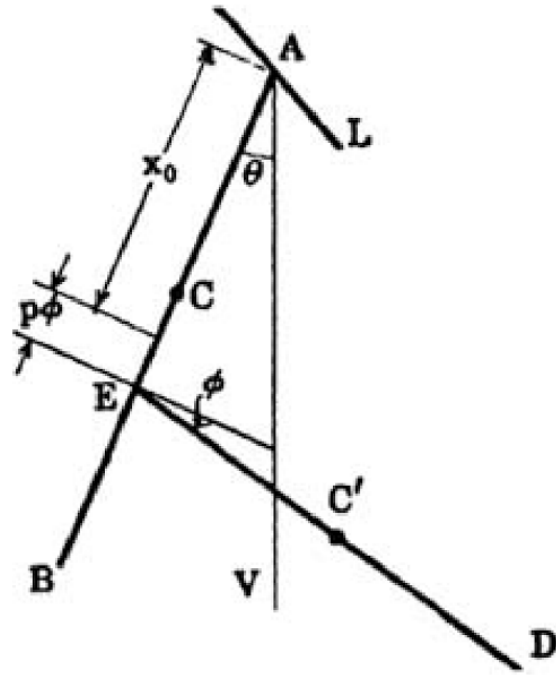


FIG. 120.—A bar  $ED$  is threaded on a shaft  $AB$ , which can turn about a horizontal axis  $L$ .

$DE$  close to II. For this position,  $\tan \theta$  is finite, since  $p$  and  $\phi$  are small of the same order. Another position occurs near  $\phi = \frac{1}{2}\pi$ ; for this,  $\tan \theta$  is small, and so  $AB$  is nearly vertical. If  $p$  is large, the right-hand side of (10.723) may exceed  $Wb^2$  for all  $\phi$ ; in this case, there is no position of equilibrium, and the bar  $DE$  simply runs down the shaft  $AB$ , turning as it goes.

*Example 4.* The last two examples considered above are three-dimensional in character. We conclude with an application of the methods of work and energy to a two-dimensional system with many degrees of freedom.

Consider a chain of  $n$  equal uniform rods, smoothly jointed together and suspended from one end  $A_1$ . (Figure 121 shows the case  $n = 5$ ). A horizontal force  $\mathbf{P}$  is applied to the other end  $A_{n+1}$  of this chain. It is required to find the equilibrium configuration.

from a position of equilibrium. Hence, equating to zero the brackets on the right of (10.725), we find

$$(10.726) \quad \left\{ \begin{array}{l} \tan \theta_1 = \frac{2P}{w(2n-1)}, \\ \tan \theta_2 = \frac{2P}{w(2n-3)}, \\ \dots \dots \dots \\ \tan \theta_n = \frac{2P}{w}. \end{array} \right.$$

These equations give the inclinations of the rods to the downward vertical in the equilibrium configuration, the tangents form a harmonic progression.

## 10.8. SUMMARY OF STATICS IN SPACE

### I. Conditions of equilibrium.

(a) For a single particle (necessary and sufficient):

$$(10.801) \quad \mathbf{P} = \mathbf{0},$$

or

$$(10.802) \quad X = 0, \quad Y = 0, \quad Z = 0.$$

(b) For any system (necessary), or for a rigid body (necessary and sufficient):

$$(10.803) \quad \mathbf{F} = \mathbf{0}, \quad \mathbf{G} = \mathbf{0}.$$

( $\mathbf{F}$  = total force,  $\mathbf{G}$  = total moment.)

(c) For any system with workless constraints (necessary and sufficient):

$$(10.804) \quad \delta W = 0, \quad (\delta W = \text{work done by applied forces}).$$

### II. Equipollence.

(a) Conditions of equipollence:

$$(10.805) \quad \mathbf{F} = \mathbf{F}', \quad \mathbf{G} = \mathbf{G}'.$$

(b) Any system of forces can be reduced to a force  $\mathbf{F}$  at an assigned point, together with a couple  $\mathbf{G}$ . If  $\mathbf{G} = p\mathbf{F}$ , the reduced system is a wrench.

### III. Displacements of a rigid body.

(a) Finite displacements:

(i) Any displacement of a rigid body with a fixed point is equivalent to a rotation  $\mathbf{n}$  (Euler's theorem).

(ii) A general displacement is equivalent to a translation  $\mathbf{s}$ , followed by rotation  $\mathbf{n}$ .

(b) Infinitesimal displacements:

(i) Infinitesimal rotations compound vectorially. The order of application is immaterial.

(ii) For a rigid body with a fixed point, the displacement of a particle of the body is

$$(10.806) \quad \delta \mathbf{n} \times \mathbf{r}.$$

(iii) In general, the displacement of a particle of the body is

$$(10.807) \quad \delta \mathbf{s} + \delta \mathbf{n} \times \mathbf{r}.$$

### IV. Work and potential energy.

(a) Work done on a particle:

$$(10.808) \quad \delta W = \mathbf{P} \cdot \delta \mathbf{s} = X \delta x + Y \delta y + Z \delta z.$$

(b) Work done on a rigid body:

$$(10.809) \quad \delta W = \mathbf{F} \cdot \delta \mathbf{s} + \mathbf{G} \cdot \delta \mathbf{n}.$$

(c) Work done on a general system:

$$(10.810) \quad \delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_n \delta q_n.$$

(d) Work done on a conservative system:

$$(10.811) \quad \delta W = -\delta V = -\sum_{r=1}^n \frac{\partial V}{\partial q_r} \delta q_r.$$

### EXERCISES X

1. A force with components  $(-7, 4, -5)$  acts at the point  $(2, 4, -3)$ . Find its moment about the origin. Find also its moment about the line



## Tangential and normal components of velocity and acceleration.

Figure 122 shows the path  $C$  of a moving particle  $A$ ;  $A_0$  is a fixed point on  $C$ . The arc length  $AA_0$  is denoted by  $s$ . From (11.101), we see that the vector  $d\mathbf{r}/ds$  has components  $dx/ds$ ,  $dy/ds$ ,  $dz/ds$  along  $\mathbf{I}$ ,  $\mathbf{J}$ ,  $\mathbf{K}$ ; it is the unit tangent vector to  $C$  at  $A$  and will be denoted by  $\mathbf{i}$ .

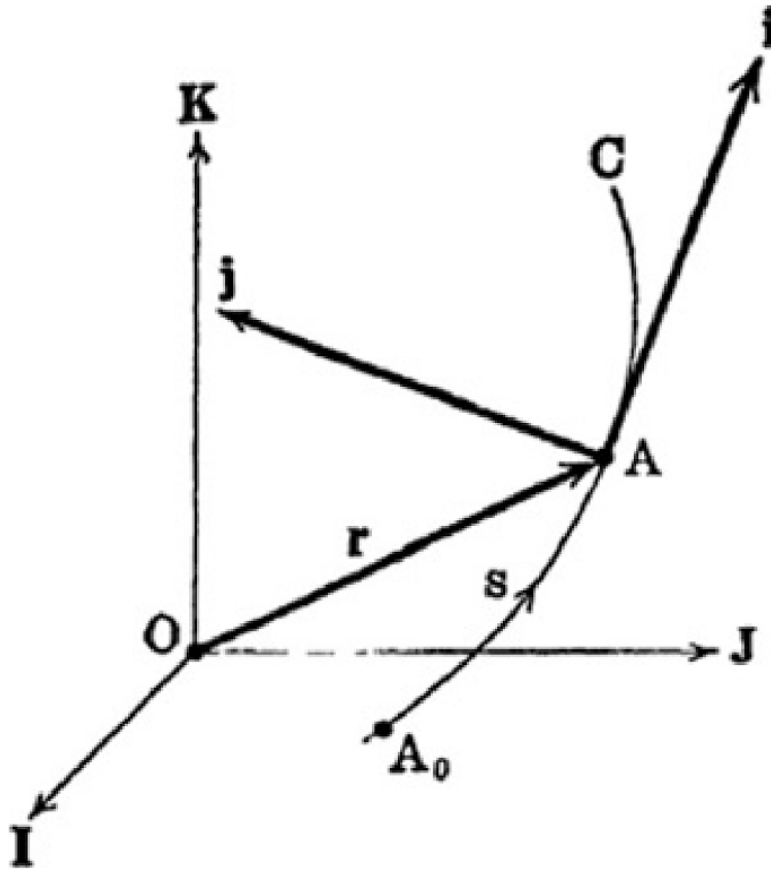


FIG. 122.—A particle moving in space.

For the velocity of  $A$  we have

$$(11.102) \quad \mathbf{q} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} = \dot{s} \mathbf{i}.$$

Hence, *the velocity of a particle is directed along the tangent to its path, and has magnitude  $\dot{s}$ .*

For the acceleration we have

$$\mathbf{f} = \frac{d\mathbf{q}}{dt} = \dot{s} \mathbf{i} + \dot{s} \frac{d\mathbf{i}}{dt} = \dot{s} \mathbf{i} + \dot{s}^2 \frac{d\mathbf{i}}{ds}.$$

But, by (10.211),

$$\frac{d\mathbf{i}}{ds} = \frac{\mathbf{j}}{\rho},$$

where  $\mathbf{j}$  is the unit principal normal vector and  $\rho$  the radius of curvature of  $C$  at  $A$ . Hence,

$$(11.103) \quad \mathbf{f} = \ddot{s}\mathbf{i} + \frac{\dot{s}^2}{\rho}\mathbf{j},$$

and so we may state: *The acceleration of a particle lies in the osculating plane to its path; the components in the directions of the tangent and principal normal are  $\ddot{s}$  and  $\dot{s}^2/\rho$ , respectively.* These results should be compared with those for the corresponding two-dimensional case (cf. Sec. 4.1).

### Components of velocity and acceleration in cylindrical coordinates.

In Fig. 123,  $A$  is the position of a particle at time  $t$  and  $M$  the foot of the perpendicular from  $A$  on the plane  $Oxy$ . The polar coordinates  $(R, \phi)$  of  $M$ , together with the  $z$ -coordinate of  $A$ , are the cylindrical coordinates  $(R, \phi, z)$  of  $A$ . Let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be unit vectors at  $A$  in the directions of the parametric lines of these coordinates (i.e., those directions in each of which just one of the three coordinates  $R, \phi, z$  increases, the other two remaining constant). We wish to find the components of the velocity and acceleration of  $A$  along  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

The vector  $\mathbf{k}$  is constant in magnitude and direction. The directions of  $\mathbf{i}$  and  $\mathbf{j}$  do not depend on  $R$  and  $z$ ; they are, however, dependent on  $\phi$ . As in Sec. 4.1 (where  $r, \theta$  correspond to  $R, \phi$ ), we have

$$(11.104) \quad \frac{d\mathbf{i}}{d\phi} = \mathbf{j}, \quad \frac{d\mathbf{j}}{d\phi} = -\mathbf{i}.$$

Since

$$\mathbf{r} = \overrightarrow{OA} = R\mathbf{i} + z\mathbf{k},$$

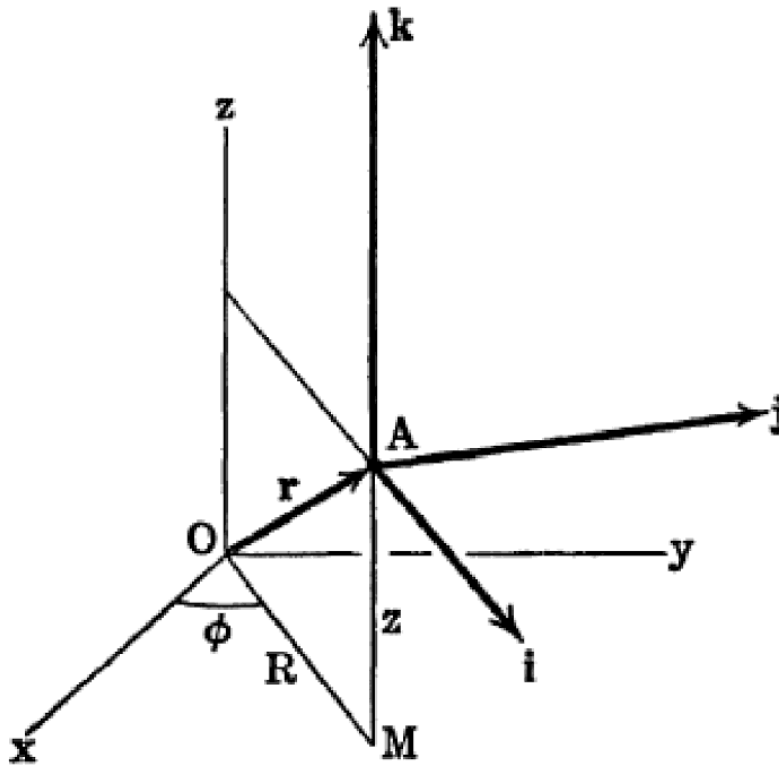


FIG. 123.—Cylindrical coordinates.

we obtain, on differentiating  $\mathbf{r}$  with respect to  $t$  and using (11.104),

$$(11.105) \quad \mathbf{q} = \frac{d\mathbf{r}}{dt} = \dot{R}\mathbf{i} + R\dot{\phi}\mathbf{j} + \dot{z}\mathbf{k}.$$

A second differentiation with respect to  $t$  gives

$$(11.106) \quad \mathbf{f} = (\ddot{R} - R\dot{\phi}^2)\mathbf{i} + \frac{1}{R} \frac{d}{dt} (R^2\dot{\phi})\mathbf{j} + \ddot{z}\mathbf{k}.$$

From these equations, we can read off the components of the velocity ( $\mathbf{q}$ ) and the acceleration ( $\mathbf{f}$ ) in the directions of  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , when required.

### Composition of velocities and accelerations.

We often need to connect the velocities (or accelerations) of a particle relative to two different frames of reference,  $S$  and  $S'$ . We shall here think only of the case where there is no relative rotation of the frames. Let  $O$  be a point fixed in  $S$  and  $O'$  a point fixed in  $S'$ . A particle  $A$  has position vectors  $\mathbf{r} = \vec{OA}$  and  $\mathbf{r}' = \vec{O'A}$ ; they are connected by

$$(11.107) \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}',$$

where  $\mathbf{r}_0 = \vec{OO'}$ . Differentiation gives

$$(11.108) \quad \mathbf{q} = \mathbf{q}_0 + \mathbf{q}', \quad \mathbf{f} = \mathbf{f}_0 + \mathbf{f}',$$

where  $\mathbf{q}, \mathbf{f}$  = velocity and acceleration of  $A$  relative to  $S$ ,

$\mathbf{q}', \mathbf{f}'$  = velocity and acceleration of  $A$  relative to  $S'$ ,

$\mathbf{q}_0, \mathbf{f}_0$  = velocity and acceleration of  $S'$  relative to  $S$ .

The equations (11.108) give the laws of composition of velocities and accelerations.

## 11.2. KINEMATICS OF A RIGID BODY

### Motion of a rigid body with a fixed point.

Consider a rigid body constrained to rotate about a fixed point  $O$ . Let  $t_1, t_2$  be two instants; in the time interval  $t_2 - t_1$  the body receives a displacement which is equivalent (cf. Sec. 10.5) to a rotation  $\mathbf{n}$  about  $O$ . If we keep  $t_1$  fixed and let  $t_2$  approach  $t_1$ , the direction of  $\mathbf{n}$  will approach some limiting direction, which we denote by the unit vector  $\mathbf{i}$ . The ratio of the angle of rotation  $n$  to the time interval  $t_2 - t_1$  will approach a limiting value  $\omega$ . The vector  $\boldsymbol{\omega} = \omega \mathbf{i}$  is called the *angular velocity* of the body at the instant  $t_1$ . At this instant the body is rotating about a line through  $O$  in the direction of  $\boldsymbol{\omega}$ ; this line is called the *instantaneous axis of rotation*. The rate of turning is  $\omega$  radians per unit time and is a rotation in the positive sense about the instantaneous axis.

In an infinitesimal time  $dt$  the body receives an infinitesimal rotation  $\boldsymbol{\omega} dt$ ; and so the displacement of a particle of the body is, by (10.501),

$$d\mathbf{r} = \boldsymbol{\omega} dt \times \mathbf{r},$$

where  $\mathbf{r}$  is the position vector relative to  $O$ . The velocity of this particle is

$$(11.201) \quad \mathbf{q} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}.$$

This formula gives the velocity of any particle of the body in terms of the angular velocity vector  $\boldsymbol{\omega}$ . Thus, if  $\boldsymbol{\omega}$  is known as a vector function of the time, we can find the velocity of any particle at any time; in other words, the single vector function  $\boldsymbol{\omega}(t)$  suffices to describe the motion.

As the body turns about  $O$ , the instantaneous axis (determined by  $\boldsymbol{\omega}$ ) will occupy different positions in the body. Since this axis always passes through  $O$ , its locus in the body is a cone with vertex  $O$ ; it is called the *body cone* (or *polhode cone*). Similarly, the locus of the instantaneous axis in space is

another cone with vertex  $O$ ; it is called the *space cone* (or *herpolhode cone*).

A rigid body moving parallel to a fundamental plane may be regarded as a body turning about a point at infinity. In this case the body and space cones become cylinders; their intersections with the fundamental plane are the body and space centrodes of our earlier theory (cf. Sec. 4.2).

We saw in Sec. 4.2 that, in the motion of a rigid body parallel to a plane, the body centrode rolls on the space centrode. Similarly, in the motion of a rigid body with a fixed point, the body cone rolls on the space cone. To establish this result we must show that:

(i) the body cone touches the space cone;

(ii) the particles of the body on the line of contact of the cones are instantaneously at rest.

Let  $OA$  be the position of the instantaneous axis of rotation at some instant. It is a generator of the fixed space cone and also of the moving body cone. After an infinitesimal time  $dt$ , another generator  $OB$  of the body cone comes into coincidence with a generator  $OB'$  of the space cone. But the displacement in time  $dt$  is an infinitesimal rotation of magnitude  $\omega dt$  about  $OA$ , and so the angle between the planes  $OAB$ ,  $OAB'$  is an infinitesimal angle. Since these planes represent the tangent planes to the two cones along the generator  $OA$ , it follows that the tangent planes cannot cut at a finite angle; the cones must therefore touch. Since all particles of the body on the instantaneous axis  $OA$  are instantaneously at rest, the second of the above conditions is also satisfied, and the result is established.

## **The components of angular velocity in terms of the Eulerian angles.**

In Sec. 10.6 we defined the Eulerian angles  $\theta$ ,  $\phi$ ,  $\psi$ ; they describe (relative to a fixed triad  $\mathbf{I}$ ,  $\mathbf{J}$ ,  $\mathbf{K}$ ) the position of a triad of unit orthogonal vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , fixed in a rigid body turning about a point  $O$  (Fig. 118). The motion of the body is determined when  $\theta$ ,  $\phi$ ,  $\psi$  are known as functions of the time  $t$ ; but this motion can also be described by the angular velocity  $\omega(t)$ . We write

$$\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k},$$

and seek expressions for  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  in terms of  $\theta$ ,  $\phi$ ,  $\psi$ , and their rates of change.

In an infinitesimal time  $dt$  the body receives the rotation  $\omega dt$ . But, by

(11.304) to

$$A'x'^2 + B'y'^2 + C'z'^2 - 2F'y'z' - 2G'z'x' - 2H'x'y' = 1.$$

The coefficients  $A'$ ,  $B'$ ,  $C'$ ,  $F'$ ,  $G'$ ,  $H'$  give the moments and products of inertia for the new axes.

The quadric represented by the equation (11.304) sums up the inertial properties of the system with respect to axes through the origin. The form of the equation changes (in the sense that the values of the coefficients change) when we rotate the coordinate axes, but the quadric itself remains an invariant model of the inertial properties. The representation of physical properties by means of a quadric surface is of frequent occurrence—it is used in elasticity, hydrodynamics, and other branches of applied mathematics. Since the coefficients in the equation of the quadric change when we change the axes, they cannot be called scalars, in the sense that mass is a scalar. Nor are they components of a vector. The whole set of six coefficients, or more precisely the array

$$(11.305) \quad \begin{array}{ccc} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{array}$$

is called a *tensor*. This is the simplest example of the concept which has played such an important part in the theory of relativity.

An array is also called a *matrix*. Just as we use a single letter to denote a vector (which may be regarded as a matrix with three elements), so we may denote a matrix by a single letter. The operations of algebra may be applied to matrices, yielding a compact and powerful notation in mechanics.\*

## Existence of principal axes and moments of inertia.

Equation (11.303) gives us the moment of inertia about any line  $L$  through  $O$  in terms of the direction cosines of  $L$  and the six coefficients shown in the array (11.305). We shall now show that the number of coefficients may be reduced from six to three by making a suitable choice of the axes  $Oxyz$ .

Let  $Oxyz$  be any axes. Then  $I$ , as given by (11.303), attains its maximum value for some line  $L_1$ . Let this maximum be  $I_1$ . Let us take axes  $Ox'y'z'$  such that  $Ox'$  coincides with  $L_1$ ; we leave the directions of the other two axes unspecified for the present, except for the conditions that they shall be

impossible since  $I_2$  is a maximum.

Therefore  $F' = 0$ , and we have for any line  $L$ ,

$$I = I_1\alpha'^2 + I_2\beta'^2 + C'\gamma'^2.$$

Substituting  $\gamma'^2 = 1 - \alpha'^2 - \beta'^2$ , we get

$$I - C' = (I_1 - C')\alpha'^2 + (I_2 - C')\beta'^2 \geq 0,$$

and so the third moment of inertia  $C'$  is the least of all moments of inertia for lines through  $O$ .

We may sum up as follows, simplifying the notation: *It is always possible to choose rectangular axes  $Oxyz$  such that the moment of inertia  $I$  of a system about a line  $L$  through  $O$  is given by*

$$(11.306) \quad I = A\alpha^2 + B\beta^2 + C\gamma^2,$$

where  $\alpha, \beta, \gamma$  are the direction cosines of  $L$  relative to  $Oxyz$ . These axes are called *principal axes of inertia at  $O$* , and the moments of inertia  $A, B, C$  about them are called *principal moments of inertia*. The planes defined by the principal axes are called *principal planes*; for any pair of principal planes, the product of inertia vanishes since  $F, G$ , and  $H$  are absent from (11.306). For principal axes, the equation of the momental ellipsoid is

$$(11.307) \quad Ax^2 + By^2 + Cz^2 = 1.$$

*Exercise.* Find principal axes of inertia for a thin straight uniform rod at its middle point.

## General method of finding principal axes and moments of inertia.

The principal axes and moments of inertia have been shown to exist. We shall now show how to find them, starting from general axes  $Oxyz$  with moments and products of inertia  $A, B, C, F, G, H$ . Let  $Ox'y'z'$  be the principal axes and  $A', B', C'$  the principal moments of inertia. Any point  $P$  has two sets of coordinates,  $(x, y, z)$  and  $(x', y', z')$ , according to the axes which are used. One set of coordinates are linear functions of the other, such that

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

for every point  $P$ . Also, for every point  $P$ , we have



$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = A'x'^2 + B'y'^2 + C'z'^2,$$

since each side represents the moment of inertia about  $OP$ , multiplied by  $OP^2$ . Therefore, no matter how the constant  $K$  is chosen, we have the identity

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy - K(x^2 + y^2 + z^2) = A'x'^2 + B'y'^2 + C'z'^2 - K(x'^2 + y'^2 + z'^2).$$

Let us denote each side of this identity by  $\Phi$ . Consider the equations

$$(11.308) \quad \frac{\partial \Phi}{\partial x'} = 0, \quad \frac{\partial \Phi}{\partial y'} = 0, \quad \frac{\partial \Phi}{\partial z'} = 0.$$

Explicitly, these equations read

$$(A' - K)x' = 0, \quad (B' - K)y' = 0, \quad (C' - K)z' = 0.$$

Rejecting the trivial solution  $x' = y' = z' = 0$ , we must choose  $K$  equal to  $A'$ ,  $B'$ , or  $C'$ . We have then the following three solutions:

$$\begin{aligned} K &= A', x' \text{ arbitrary, } y' = 0, z' = 0; \\ K &= B', x' = 0, y' \text{ arbitrary, } z' = 0; \\ K &= C', x' = 0, y' = 0, z' \text{ arbitrary.} \end{aligned}$$

Thus (11.308) have nontrivial solutions provided  $K$  is equal to one of the principal moments of inertia; the corresponding values of  $x'$ ,  $y'$ ,  $z'$  give the principal axes.

Now

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial \Phi}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial \Phi}{\partial z'} \frac{\partial z'}{\partial x},$$

and similar equations could be written for  $\partial \Phi / \partial y$  and  $\partial \Phi / \partial z$ . Therefore, (11.308) imply that

$$(11.309) \quad \frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} = 0, \quad \frac{\partial \Phi}{\partial z} = 0.$$

If  $K$ ,  $x'$ ,  $y'$ ,  $z'$  are chosen as above, (11.308) are satisfied, and therefore (11.309) are satisfied. Explicitly, (11.309) read

$$(11.310) \quad \begin{cases} (A - K)x - Hy - Gz = 0, \\ -Hx + (B - K)y - Fz = 0, \\ -Gx - Fy + (C - K)z = 0. \end{cases}$$

Since these equations have a solution other than  $x = y = z = 0$ , it follows that

$$(11.311) \quad \begin{vmatrix} A - K & -H & -G \\ -H & B - K & -F \\ -G & -F & C - K \end{vmatrix} = 0.$$

This is a cubic equation for  $K$ , and, as we have seen, its three roots are the three principal moments of inertia. To sum up: *Starting with general axes Oxyz with moments and products of inertia  $A, B, C, F, G, H$ , the three principal moments of inertia at  $O$  are the values of  $K$  satisfying the cubic determinantal equation (11.311), and the directions of the three principal axes are given by the ratios  $x:y:z$  determined by (11.310) when the above values of  $K$  are substituted.*

The problem of finding principal axes and moments of inertia is essentially the same as the geometrical problem of finding the directions and magnitudes of the principal axes of an ellipsoid from its general equation.\* The use of the equations (11.309) is most naturally suggested by the problem† of finding stationary values (including maximum and minimum values) of the expression

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy,$$

subject to the condition  $x^2 + y^2 + z^2 = 1$ .

## Method of symmetry.

For a body which exhibits symmetry, it is often possible to find principal axes of inertia very simply.

In Sec. 3.1 the idea of symmetry was used in connection with mass centers. A more thorough discussion requires the concept of a *covering operation*, which we now proceed to define.

If we rotate a body of revolution about its axis through any angle, we do not alter the distribution of matter—the whole body appears exactly as before. Similarly, if we turn a three-bladed propeller about its axis through an angle  $2\pi/3$ , the final distribution of matter is that with which we started. These rotations are examples of covering operations. In general, a *covering*

ellipsoid must be of revolution. If a rotation through an angle  $\pi$  about a line  $L$  is a covering operation, then  $L$  must be a principal axis. If a reflection in a plane  $\parallel$  is a covering operation, then  $n$  must be a principal plane.

We shall now apply these facts to the momental ellipsoid. The truth of the following statements will be obvious:

(i) An axis of  $n$ -gonal symmetry is a principal axis of inertia at any point of itself. (Example: a two-bladed propeller.)

(ii) At any point on an axis of trigonal or tetragonal symmetry, the momental ellipsoid has this axis for axis of revolution, and two of the principal moments of inertia are equal. (Example: a three- or four-bladed propeller.)

(iii) The normal to a plane of symmetry is a principal axis of inertia at the point where it cuts the plane of symmetry. (Example: the hull of a ship.)

Principal axes of inertia for a number of bodies are given in the table on page 324. In each case an argument, based on the ideas of symmetry, can be used to verify that the principal axes are given correctly.

## The momental ellipse.

Let us now consider a distribution of matter in a plane  $\Pi$ , and let  $Ox$ ,  $Oy$  be rectangular axes in this plane. Since  $\Pi$  is a plane of symmetry, its normal at  $O$  is a principal axis of inertia, and the section of the momental ellipsoid at  $O$  by the plane  $\Pi$  is a principal section; it is called the *momental ellipse* at  $O$ .

If  $A$  and  $B$  denote the moments of inertia about  $Ox$ ,  $Oy$ , respectively, and  $H$  denotes the product of inertia with respect to planes through  $Ox$ ,  $Oy$ , perpendicular to  $\Pi$ , the equation of this ellipse is

$$(11.312) \quad Ax^2 - 2Hxy + By^2 = 1.$$

(We see this by introducing the third axis  $Oz$  and putting  $z = 0$  in the equation of the momental ellipsoid.) It is clear that the principal axes of this ellipse are principal axes of inertia at  $O$ . To find them we proceed as follows.

Let  $Ox'$ ,  $Oy'$  be new axes at  $O$ ,  $Ox'$  making an angle  $\theta$  with  $Ox$ . If  $(x', y')$ ,  $(x, y)$  denote the coordinates of a point referred to the axes  $Ox'y'$ ,  $Oxy$ , respectively, then

$$x = x' \cos \theta - y' \sin \theta, \quad y = x' \sin \theta + y' \cos \theta.$$

where  $n$  is the number of particles in the system.

We note that, if  $O$  is fixed in the frame of reference, then  $\mathbf{q}_i = \dot{\mathbf{r}}_i$ . In that case the components of  $\mathbf{h}$  along rectangular axes fixed in the frame are

$$(11.503) \quad \sum_{i=1}^n m_i(y_i \dot{z}_i - z_i \dot{y}_i), \quad \sum_{i=1}^n m_i(z_i \dot{x}_i - x_i \dot{z}_i), \\ \sum_{i=1}^n m_i(x_i \dot{y}_i - y_i \dot{x}_i).$$

Let us now consider the effect of changing the frame of reference. Let  $S'$  be a new frame, having a velocity  $\mathbf{q}_0$  of translation relative to  $S$ . Then the velocities  $\mathbf{q}_i, \mathbf{q}'_i$  of a particle relative to  $S, S'$ , respectively, are connected by

$$(11.504) \quad \mathbf{q}_i = \mathbf{q}_0 + \mathbf{q}'_i,$$

according to (11.108). The angular momenta about  $O$  are then

$$\mathbf{h} = \sum_{i=1}^n (\mathbf{r}_i \times m_i \mathbf{q}_i) \text{ for } S; \quad \mathbf{h}' = \sum_{i=1}^n (\mathbf{r}_i \times m_i \mathbf{q}'_i) \text{ for } S'.$$

Substituting from (11.504) in the expression for  $\mathbf{h}$ , we find

$$(11.505) \quad \mathbf{h} = \left( \sum_{i=1}^n m_i \mathbf{r}_i \right) \times \mathbf{q}_0 + \mathbf{h}'.$$

If  $O$  is the mass center,  $\sum_{i=1}^n m_i \mathbf{r}_i = \mathbf{0}$ , and so the first term on the right vanishes. This gives the following remarkable result: *Angular momentum about the mass center is the same for all frames of reference in relative translational motion.* Generally it is most convenient to use a frame of reference in which the mass center is fixed.

In speaking of angular momentum about a point  $O$ , we shall in future always understand a frame of reference in which  $O$  is fixed.

## Angular momentum of a rigid body.

The most interesting application of (11.502) is to the case of a rigid body turning about  $O$ . The formulas which we are about to develop are fundamental in gyroscopic theory.

In a slightly different notation, we have, for the angular momentum about  $O$ ,

along this axis. Then  $\omega_1 = 0$ ,  $\omega_2 = 0$ ,  $\omega_3 = \omega$ , and (11.508) gives

$$(11.510) \quad h_1 = -G\omega, \quad h_2 = -F\omega, \quad h_3 = C\omega.$$

Thus the angular momentum vector does not lie along the axis of rotation, unless the latter is a principal axis of inertia. However, the component  $h_3$  along the axis of rotation is equal to the product of the moment of inertia about that axis and the angular velocity. This is in agreement with (7.117).

## 11.6. SUMMARY OF KINEMATICS, KINETIC ENERGY, AND ANGULAR MOMENTUM

### I. Kinematics of a particle.

Velocity:

$$(11.601) \quad \mathbf{q} = \frac{d\mathbf{r}}{dt} = s\mathbf{i}, \quad (\mathbf{i} = \text{unit tangent vector}).$$

Acceleration:

$$(11.602) \quad \mathbf{f} = \frac{d\mathbf{q}}{dt} = s\mathbf{i} + \frac{s^2}{\rho}\mathbf{j}, \quad (\mathbf{j} = \text{unit principal normal vector}).$$

### II. Kinematics of a rigid body.

(a) Rigid body with a fixed point:

Motion described by vector  $\omega$ ; velocity of any particle of the body is

$$(11.603) \quad \mathbf{q} = \omega \times \mathbf{r}.$$

(b) Rigid body in general motion:

Motion described by vectors  $\mathbf{q}_A$ ,  $\omega$ ; velocity of any particle of the body is

$$(11.604) \quad \mathbf{q} = \mathbf{q}_A + \omega \times \mathbf{r}.$$

### III. Moments and products of inertia.

(a) General formulas:

$$(12.103) \quad m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z,$$

where  $X, Y, Z$  are the components of  $\mathbf{P}$  along the axes. These are the equations of motion of a particle in rectangular Cartesians; other forms of the equations of motion are obtained below.

Let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be unit vectors along the tangent, principal normal and binormal to the path of the particle. By (11.103),

$$\mathbf{f} = \dot{s}\mathbf{i} + \frac{\dot{s}^2}{\rho}\mathbf{j},$$

where  $s$  denotes arc length along the path and  $\rho$  is the radius of curvature. Writing

$$\mathbf{P} = P_1 \mathbf{i} + P_2 \mathbf{j} + P_3 \mathbf{k},$$

we obtain from (12.101) the following *intrinsic equations of motion*:

$$(12.104) \quad m\ddot{s} = P_1, \quad \frac{m\dot{s}^2}{\rho} = P_2, \quad 0 = P_3.$$

Now let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be unit vectors in the directions of the parametric lines of cylindrical coordinates  $(R, \phi, z)$ . By (11.106), we have

$$\mathbf{f} = (\ddot{R} - R\dot{\phi}^2)\mathbf{i} + \frac{1}{R} \frac{d}{dt}(R^2\dot{\phi})\mathbf{j} + \ddot{z}\mathbf{k}.$$

Thus, if

$$\mathbf{P} = P_R \mathbf{i} + P_\phi \mathbf{j} + P_z \mathbf{k},$$

we obtain, as *equations of motion in cylindrical coordinates*,

$$(12.105) \quad m(\ddot{R} - R\dot{\phi}^2) = P_R, \quad m \frac{1}{R} \frac{d}{dt}(R^2\dot{\phi}) = P_\phi, \quad m\ddot{z} = P_z.$$

Equations (12.103), (12.104), and (12.105) are probably the most useful forms of the equations of motion of a particle. Other forms may be obtained by following the same general plan, namely, resolution of vectors along a suitably chosen orthogonal triad.

When the path of a particle is to be found, it is better to use some form such as (12.103) or (12.105), rather than the intrinsic equations (12.104). However, if the path is known beforehand, the equations (12.104) are particularly convenient. For example, consider a particle sliding down a smooth curve

under gravity. In this case  $P_1$  is simply the component of the particle's weight in the direction of the tangent and is therefore known. Integration of the first equation in (12.104) gives  $\dot{s}$  (and  $s$ ) in terms of the time. The other two equations then give the reaction of the curve on the particle without further integration.

There is a point of interest in connection with (12.104). From the last of these equations, it is clear that the path is such that the osculating plane contains the applied force. We recall that, for a flexible cable in equilibrium [cf. (10.217)], the osculating plane also contains the external force; there is a close analogy between the two problems.

*Exercise.* A particle moves on a smooth surface under no forces except the reaction of the surface. Show that its path is a geodesic on the surface. Is this result true when the surface is rough?

## Integration of the equations of motion.

To obtain the equations of motion of a particle is one question, but to solve them is another. The second task is much harder than the first. Indeed, we may say that only a very few of all possible problems in dynamics can be completely solved, if by *solution* we mean the expression of the coordinates as easily calculable functions of the time  $t$ . However, it is always possible to obtain solutions in the form of power series in  $t$ . Consideration of this process leads to the following important general theorem: *The motion of a particle is determined when its initial position and velocity are given.*

Let us sketch the proof of this theorem in the case of a free particle, moving in accordance with the equations (12.103). We shall suppose that  $X, Y, Z$  are given functions of  $x, y, z$  and perhaps of  $\dot{x}, \dot{y}, \dot{z}, t$ , also. (Consider, for example, a projectile under the action of gravity and air resistance, as in Sec. 6.2.) Then the equations (12.103) give  $x_0, \dot{x}_0, \ddot{x}_0$  in terms of

$$(12.106) \quad x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0,$$

the subscript indicating evaluation at  $t = 0$ . If we differentiate (12.103) and consider the resulting equations at  $t = 0$ , we see that they give the third derivatives of  $x, y, z$  with respect to  $t$  at  $t = 0$  in terms of the quantities (12.106), since  $\dot{x}_0, \dot{y}_0, \dot{z}_0$  have already been found. Proceeding in this way, we can determine *all* derivatives of  $x, y, z$  at  $t = 0$  in terms of the quantities (12.106). Thus, we have all the coefficients in the following Taylor expansions for  $x, y, z$ :



$$\begin{aligned}
 (12.108) \quad \dot{\mathbf{h}} &= \dot{\mathbf{r}} \times m\mathbf{q} + \mathbf{r} \times m\dot{\mathbf{q}} \\
 &= \mathbf{q} \times m\mathbf{q} + \mathbf{r} \times m\mathbf{f} \\
 &= \mathbf{r} \times \mathbf{P},
 \end{aligned}$$

where  $\mathbf{P}$  is the force acting on the particle. In words, *the rate of change of angular momentum of a particle about a fixed point is equal to the moment of the applied force about that point.*

If we take  $O$  as origin of rectangular axes  $Oxyz$  and resolve vectors in the directions of these axes, we obtain

$$(12.109) \quad \begin{cases} m(y\dot{z} - z\dot{y}) = yZ - zY, \\ m(z\dot{x} - x\dot{z}) = zX - xZ, \\ m(x\dot{y} - y\dot{x}) = xY - yX, \end{cases}$$

where  $X, Y, Z$  are the components of  $\mathbf{P}$ . The last of these equations is the same as that obtained in Sec. 5.1 for a particle moving in the plane  $Oxy$ , moments being taken about  $O$  (or  $Oz$ ).

## Principle of energy.

For a moving particle we have, as in Sec. 5.1,

$$(12.110) \quad \dot{T} = \dot{W},$$

where  $\dot{T}$  is the rate of increase of the kinetic energy and  $\dot{W}$  is the rate at which the applied forces do work. This general form of the principle of energy is of little use, except in the case where the working forces are conservative. Then  $\dot{W} = -\dot{V}$ , where  $V$  is the potential energy of the particle and (12.110) gives, on integration,

$$(12.111) \quad T + V = E,$$

where  $E$  is a constant, the *total energy*.

This is a single equation for  $z$  as a function of  $t$ ; when this equation has been solved, (12.112) gives  $R$  in terms of  $t$  directly, and  $\phi$  can be found by a quadrature from (12.114). The solution of (12.115) is given in the next chapter.

## 12.2. MOTION OF A SYSTEM

### Principle of linear momentum; motion of the mass center.

We now recall some results established in Sec. 5.2. If  $m_i$  and  $\mathbf{q}_i$  denote the mass and velocity of the  $i$ th particle of a system, then the linear momentum is

$\mathbf{q}_i$  = velocity of  $i$ th particle relative to  $O$ ,

$n$  = number of particles in system.

In what follows, we shall consider  $O$  to be either a fixed point in a Newtonian frame of reference or the mass center of the system.

The rate of change of  $\mathbf{h}$  is

$$\dot{\mathbf{h}} = \sum_{i=1}^n (\dot{\mathbf{r}}_i \times m_i \mathbf{q}_i + \mathbf{r}_i \times m_i \dot{\mathbf{q}}_i).$$

Since  $\dot{\mathbf{r}}_i = \mathbf{q}_i$ , the first vector product vanishes. Thus we have

$$(12.205) \quad \dot{\mathbf{h}} = \sum_{i=1}^n (\mathbf{r}_i \times m_i \mathbf{f}_i),$$

where  $\mathbf{f}_i$  is the acceleration of the  $i$ th particle relative to  $O$ .

If  $O$  is a fixed point, then  $\mathbf{f}_i$  is acceleration relative to a Newtonian frame, and so

$$m_i \mathbf{f}_i = \mathbf{P}_i + \mathbf{P}'_i,$$

where  $\mathbf{P}_i$ ,  $\mathbf{P}'_i$  are, respectively, the external and internal force on the  $i$ th particle. Hence, by (12.205),

$$(12.206) \quad \dot{\mathbf{h}} = \sum_{i=1}^n \mathbf{r}_i \times \mathbf{P}_i + \sum_{i=1}^n \mathbf{r}_i \times \mathbf{P}'_i.$$

The second summation vanishes, since the internal forces have a moment about any point (cf. Sec. 10.2). Hence,

$$(12.207) \quad \dot{\mathbf{h}} = \mathbf{G},$$

where  $\mathbf{G}$  is the total moment of the external forces about the fixed point  $O$ .

If  $O$  is the mass center, the acceleration of the  $i$ th particle relative to a Newtonian frame  $S$  is

$$\mathbf{f}_0 + \mathbf{f}_i,$$

where  $\mathbf{f}_0$  is the acceleration of  $O$  relative to  $S$  [cf. (11.108)]. Hence, the equation of motion of the  $i$ th particle is

$$m_i (\mathbf{f}_0 + \mathbf{f}_i) = \mathbf{P}_i + \mathbf{P}'_i.$$

Substitution for  $m_i \mathbf{f}_i$  in (12.205) gives

$$(12.208) \quad \dot{\mathbf{h}} = \sum_{i=1}^n \mathbf{r}_i \times \mathbf{P}_i + \sum_{i=1}^n \mathbf{r}_i \times \mathbf{P}'_i - \left( \sum_{i=1}^n m_i \mathbf{r}_i \right) \times \mathbf{f}_0.$$

The second summation vanishes as before, and the last vanishes since  $\sum_{i=1}^n m_i \mathbf{r}_i = \mathbf{0}$ . Hence, we obtain again an equation of the form (12.207), where  $\mathbf{G}$  is now the total moment of the external forces about the mass center.

We may sum up the principle of angular momentum as follows: *The rate of change of the angular momentum of a system about a point, either fixed or moving with the mass center, is equal to the total moment of the external forces about that point; in symbols,*

$$(12.209) \quad \dot{\mathbf{h}} = \mathbf{G}.$$

The equations (12.203) and (12.209) are fundamental in dynamics and, indeed, in statics as well. Like the general conditions of equilibrium  $\mathbf{F} = \mathbf{0}$ ,  $\mathbf{G} = \mathbf{0}$ , they hold for any system—it may be the whole, or any part, of a given distribution of matter. When the system is a single rigid body, (12.203) and (12.209) provide two vector equations for  $\mathbf{q}_0$  and  $\omega$ , the velocity of a base point (the mass center) and the angular velocity of the body (cf. Sec. 11.2). Moreover, when applied to a system in equilibrium, for which  $\dot{\mathbf{q}}_0$  and  $\dot{\mathbf{h}}$  vanish, they reduce to the conditions (10.207), the basic equations in statics.

*Example.* As a simple illustration, let us consider a cylinder rolling down an inclined plane. The mass center moves in a vertical plane, and so the vectors  $\dot{\mathbf{q}}_0$  and  $\mathbf{F}$  lie in this plane. Resolving them along and perpendicular to the inclined plane, we obtain the first two equations in (7.312). The angular velocity  $\omega$  is parallel to the axis of the cylinder, and the angular momentum about the mass center is

$$\mathbf{h} = I\omega,$$

where  $I$  is the moment of inertia about the axis of the cylinder. Since  $\mathbf{h}$  has a fixed direction, (12.209) gives a single scalar equation—the third and last of the equations (7.312).

## Principle of energy.

In addition to the principles of linear and angular momentum, there is a third general principle—the principle of energy. This principle, established in Sec. 5.2, is very useful when the working forces are conservative. Then it leads to the *law of conservation of energy*,

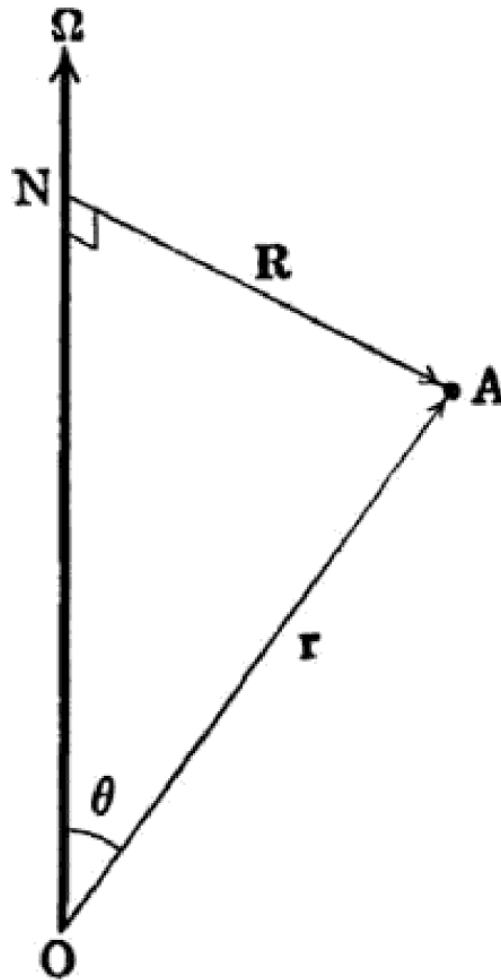


FIG. 127.—The vectors  $\mathbf{R}$  and  $\mathbf{r}$  for a particle  $A$ .

$$\boldsymbol{\Omega} \cdot \mathbf{r} = \Omega r \cos \theta,$$

where  $\theta$  is the angle  $AON$ . The acceleration of transport is, therefore,

#### 12.4. MOTION OF A RIGID BODY

The general principles of Sec. 12.2 govern the motion of any system. In this section, they are used to find explicit equations of motion for a rigid body.

##### **Rigid body with a fixed point.**

Consider a rigid body constrained to rotate about a fixed point  $O$ . By (11.509), the angular momentum about  $O$  is

$$(12.401) \quad \mathbf{h} = A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k},$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  = unit vectors in the directions of principal axes of inertia at  $O$ ,

$A, B, C$  = principal moments of inertia at  $O$ ,

$\omega_1, \omega_2, \omega_3$  = components of the angular velocity  $\omega$  of the body in the

directions  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

As pointed out in Sec. 11.4, in general the principal axes at  $O$  are fixed in the body; in that case the triad  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  has the angular velocity  $\omega$ . But if  $A, B, C$  are not all different, we may use a principal triad which is fixed neither in the body nor in space. To allow for all possibilities, we shall denote the angular velocity of the triad by  $\Omega$ , noting that  $\Omega = \omega$  if the triad is fixed in the body.

Writing

$$\Omega = \Omega_1 \mathbf{i} + \Omega_2 \mathbf{j} + \Omega_3 \mathbf{k},$$

we apply (12.306) and obtain

$$\begin{aligned} (12.402) \quad \dot{\mathbf{h}} &= \frac{\delta \mathbf{h}}{\delta t} + \Omega \times \mathbf{h} \\ &= A\dot{\omega}_1 \mathbf{i} + B\dot{\omega}_2 \mathbf{j} + C\dot{\omega}_3 \mathbf{k} + (\Omega_1 \mathbf{i} + \Omega_2 \mathbf{j} + \Omega_3 \mathbf{k}) \\ &\quad \times (A\omega_1 \mathbf{i} + B\omega_2 \mathbf{j} + C\omega_3 \mathbf{k}) \\ &= (A\dot{\omega}_1 - B\omega_2 \Omega_3 + C\omega_3 \Omega_2) \mathbf{i} \\ &\quad + (B\dot{\omega}_2 - C\omega_3 \Omega_1 + A\omega_1 \Omega_3) \mathbf{j} \\ &\quad + (C\dot{\omega}_3 - A\omega_1 \Omega_2 + B\omega_2 \Omega_1) \mathbf{k}. \end{aligned}$$

Now, by (12.209),

$$\dot{\mathbf{h}} = \mathbf{G},$$

where  $\mathbf{G}$  is the total moment of the external forces about  $O$ ; hence (12.402) gives, as the *equations of motion of a rigid body with a fixed point*,

$$(12.403) \quad \begin{cases} A\dot{\omega}_1 - B\omega_2 \Omega_3 + C\omega_3 \Omega_2 = G_1, \\ B\dot{\omega}_2 - C\omega_3 \Omega_1 + A\omega_1 \Omega_3 = G_2, \\ C\dot{\omega}_3 - A\omega_1 \Omega_2 + B\omega_2 \Omega_1 = G_3, \end{cases}$$

where  $G_1, G_2, G_3$  are the components of  $\mathbf{G}$  along  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

If  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are fixed in the body, so that  $\Omega = \omega$ , the equations (12.403) become

$$(12.404) \quad \begin{cases} A\dot{\omega}_1 - (B - C)\omega_2 \omega_3 = G_1, \\ B\dot{\omega}_2 - (C - A)\omega_3 \omega_1 = G_2, \\ C\dot{\omega}_3 - (A - B)\omega_1 \omega_2 = G_3. \end{cases}$$

These are *Euler's equations of motion for a rigid body with a fixed point*.

When the working forces are conservative, we can use, in place of any one of the three equations in (12.403) or (12.404), the following equation, deduced from the principle of energy (12.210):

$$(12.405) \quad \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + V = E.$$

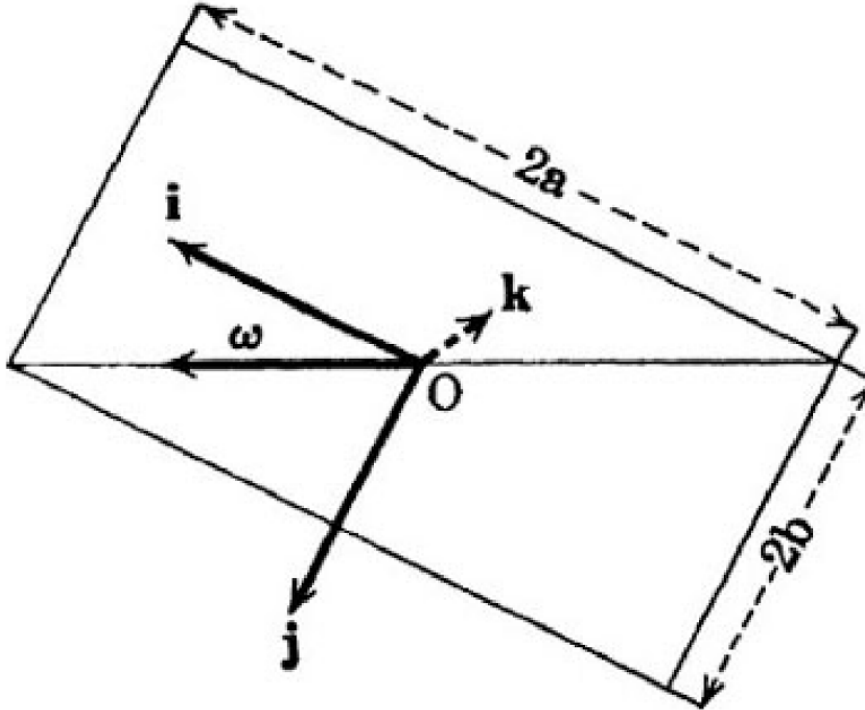


FIG. 128.—A rectangular plate spinning about a diagonal.

*Example 1. A rectangular plate spins with constant angular velocity  $\omega$  about a diagonal. Find the couple which must act on the plate in order to maintain this motion.*

In Fig. 128,  $O$  is the mass center of the plate and  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  are unit vectors along the principal axes of inertia at  $O$ ;  $\mathbf{k}$  is normal to the plate, and  $\mathbf{i}$  and  $\mathbf{j}$  lie in its plane,  $\mathbf{i}$  being parallel to the length. The principal moments of inertia at  $O$  are

$$(12.406) \quad A = \frac{1}{3}mb^2, \quad B = \frac{1}{3}ma^2, \quad C = \frac{1}{3}m(a^2 + b^2),$$

where  $m$  is the mass of the plate,  $2a$  its length, and  $2b$  its breadth.

If  $\alpha$  is the angle between  $\mathbf{i}$  and the axis of rotation, so that  $\tan \alpha = b/a$ , the angular velocity of the plate is

$$\omega = \omega \cos \alpha \mathbf{i} + \omega \sin \alpha \mathbf{j}.$$

The components of  $\omega$  in the directions  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  are, therefore,

$$\omega_1 = \omega \cos \alpha, \quad \omega_2 = \omega \sin \alpha, \quad \omega_3 = 0.$$

Substituting these values of  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  and the values of  $A$ ,  $B$ ,  $C$  from (12.406) in the

equations (12.404), we get

$$(12.407) \quad G_1 = 0, \quad G_2 = 0, \quad G_3 = \frac{1}{2}m(a^2 - b^2)\omega^2 \sin \alpha \cos \alpha \\ = \frac{1}{2}m\omega^2 ab \frac{a^2 - b^2}{a^2 + b^2}.$$

These are the components of the couple  $\mathbf{G}$  which must act on the plate; we observe that the axis of the couple is normal to the plate and turns with it.

If we suppose the plate to turn in bearings at the ends of the fixed diagonal and to be subject only to the reactions at these bearings, then clearly it is these reactions which supply the couple  $\mathbf{G}$ . Each reaction lies in the plane of the plate and is of magnitude

$$\frac{1}{6}m\omega^2 ab \frac{a^2 - b^2}{(a^2 + b^2)^{\frac{1}{2}}}.$$

As the plate turns, these reactions turn with it.

Fluctuating reactions of this sort must be avoided in the case of flywheels and rotors. It is not enough to make sure that the mass center lies on the axis of rotation. The rotating body must be balanced so that the axis of rotation is a principal axis of inertia. It is left to the reader to prove, by means of (12.404), that the fluctuating reactions vanish for any rotating body if, and only if, this condition is satisfied.

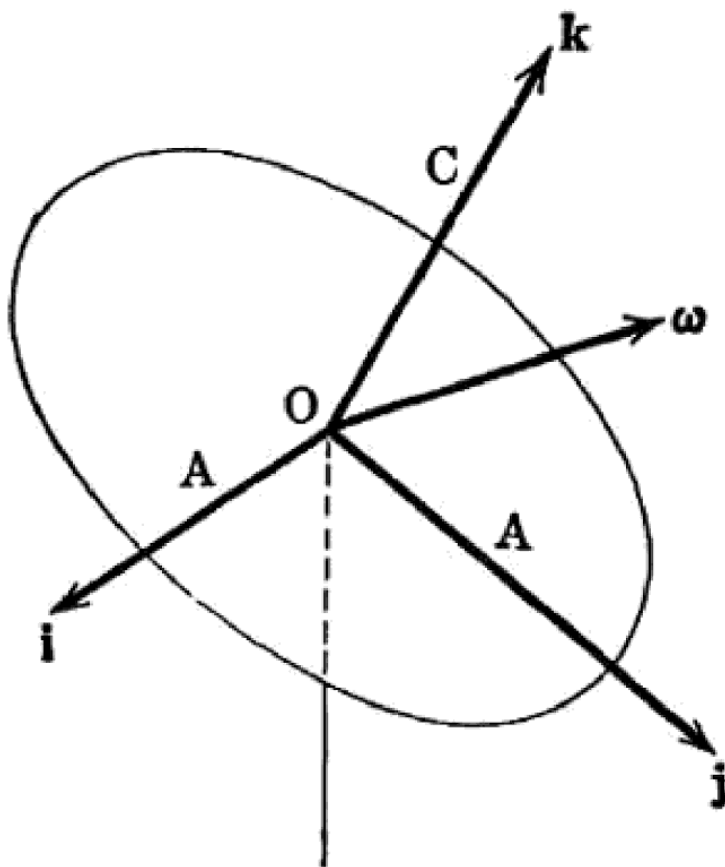


FIG. 129.—Disk spinning about its center,  $\mathbf{i}$  and  $\mathbf{j}$  fixed in the disk.

*Example 2. A circular disk of radius  $a$  and mass  $m$  is supported on a needle point at its*



center; it is set spinning with angular velocity  $\omega_0$  about a line making an angle  $\alpha$  with the normal to the disk. Find the angular velocity of the dish at any subsequent time.

In Fig. 129,  $\mathbf{k}$  is a unit vector normal to the disk at the center  $O$ , and  $\mathbf{i}, \mathbf{j}$  are fixed in the plane of the disk; we suppose  $\mathbf{j}$  chosen so that the initial angular velocity lies in the plane of  $\mathbf{k}$  and  $\mathbf{j}$ . The angular velocity of the disk at any time is

$$\omega = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k};$$

at  $t = 0$ ,

$$\omega_1 = 0, \omega_2 = \omega_0 \sin \alpha, \omega_3 = \omega_0 \cos \alpha.$$

The principal moments of inertia at  $O$  are

$$A = B = \frac{1}{2}ma^2, \quad C = \frac{1}{2}ma^2.$$

Since the external forces (the reaction at  $O$  and the weight of the disk) have no moment about  $O$ , the equations (12.404) give

$$(12.408) \quad \begin{cases} A\dot{\omega}_1 - (A - C)\omega_2\omega_3 = 0, \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 = 0, \\ C\dot{\omega}_3 = 0. \end{cases}$$

From the last of these equations it follows that  $\omega_3$  is constant; hence,  $\omega_3 = \omega_0 \cos \alpha$ . Multiplying the second equation in (12.408) by  $i (= \sqrt{-1})$  and adding the result to the first of these equations, we get

$$A\dot{\xi} - i(C - A)\omega_0 \cos \alpha \xi = 0,$$

where  $\xi = \omega_1 + i\omega_2$ . Since  $C = 2A$ , this equation can be written

$$\dot{\xi} - i\omega_0 \cos \alpha \xi = 0;$$

the general solution is

$$\xi = \xi_0 e^{i\omega_0 t \cos \alpha},$$

where  $\xi_0$  is a constant. From the initial conditions, we have

$$\xi_0 = i\omega_0 \sin \alpha,$$

and hence

$$(12.409) \quad \omega_1 = -\omega_0 \sin \alpha \sin(\omega_0 t \cos \alpha), \quad \omega_2 = \omega_0 \sin \alpha \cos(\omega_0 t \cos \alpha), \\ \omega_3 = \omega_0 \cos \alpha.$$

This is the solution of the problem as stated, but it does not tell us at once how the disk moves in space. To find this, we must either introduce the Eulerian angles defining the positions of  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  relative to fixed axes, or use a different method.

*Example 3. Find the motion in space of the disk considered in Example 2.*

In Fig. 130,  $\mathbf{h}$  is the angular momentum vector,  $\boldsymbol{\omega}$  the angular velocity vector, and  $\mathbf{k}$  a unit vector normal to the disk as before;  $\mathbf{i}$  and  $\mathbf{j}$  are unit vectors in the plane of the disk, but not fixed in it. The vector  $\mathbf{j}$  is taken in the plane determined by  $\mathbf{h}$  and  $\mathbf{k}$ . We note the following facts:

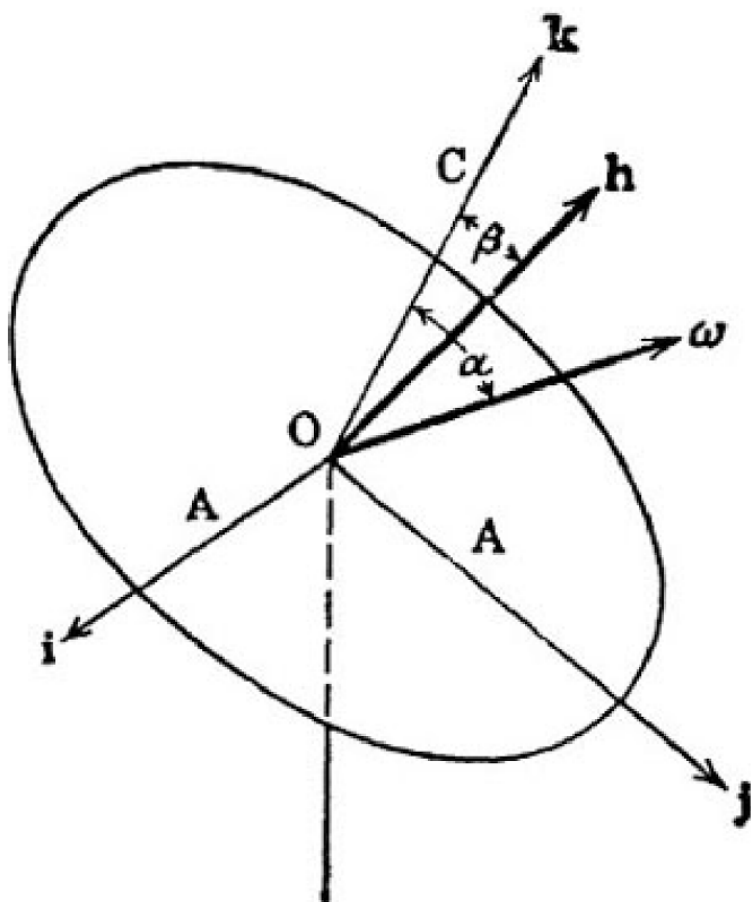


FIG. 130.—Disk spinning about its center;  $\mathbf{j}$  coplanar with  $\mathbf{h}$  and  $\mathbf{k}$ .

(i) Since the external forces have no moment about  $O$ , then, by (12.209),  $\mathbf{h}$  is a constant vector—it has a fixed direction in space determined by the initial conditions.

(ii) Since  $\mathbf{h} = A\omega_1\mathbf{i} + A\omega_2\mathbf{j} + C\omega_3\mathbf{k}$  and  $h_1 = 0$ , then  $\omega_1 = 0$  and  $\boldsymbol{\omega}$  lies in the plane of  $\mathbf{j}$  and  $\mathbf{k}$ .

(iii) Since the triad  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  is not fixed in the disk, its angular velocity  $\boldsymbol{\Omega}$  is different from  $\boldsymbol{\omega}$ . However,  $\mathbf{k}$  is fixed both in the triad and in the disk. As a point of the triad, the extremity of  $\mathbf{k}$  has velocity  $\boldsymbol{\Omega} \times \mathbf{k}$ ; as a point of the disk, it has velocity  $\boldsymbol{\omega} \times \mathbf{k}$ . Hence,

$$(\Omega_1\mathbf{i} + \Omega_2\mathbf{j} + \Omega_3\mathbf{k}) \times \mathbf{k} = (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) \times \mathbf{k},$$

and so  $\Omega_1 = \omega_1$ ,  $\Omega_2 = \omega_2$ . Since  $\omega_1 = 0$ , we have  $\Omega_1 = 0$ .

Applying the general equations (12.403) and making use of the above facts, we get

$$\begin{aligned} -A\omega_2\Omega_3 + C\omega_3\omega_2 &= 0, \\ A\dot{\omega}_2 &= 0, \\ C\dot{\omega}_3 &= 0. \end{aligned}$$

Thus,  $\omega_2$ ,  $\omega_3$ ,  $h_2$ ,  $h_3$  are constants, and  $\Omega_3/\Omega_2 = \Omega_1/\omega_2 = C\omega_3/A\omega_2 = h_3/h_2$ ; the angular velocity  $\Omega$  of the triad has constant magnitude and lies along the fixed direction  $\mathbf{h}$ . The following facts concerning the motion are now obvious:

(i) The disk spins about its normal  $\mathbf{k}$  at a constant rate  $\omega_3$ .

(ii) The angle  $\beta$  between  $\mathbf{k}$  and  $\mathbf{h}$ , given by  $h \cos \beta = h_3$ , is constant; the normal to the disk moves on a cone with axis  $\mathbf{h}$ , turning about  $\mathbf{h}$  at the constant rate  $\Omega$ .

(iii) The angle  $\alpha$  between  $\omega$  and  $\mathbf{k}$ , given by  $\omega \cos \alpha = \omega_3$ , is constant; the angular velocity vector  $\omega$  describes a cone about the normal to the disk. This is the body cone (cf. Sec. 11.2). The angle  $\alpha - \beta$  between  $\omega$  and  $\mathbf{h}$  is also constant, and so the space cone has constant semivertical angle  $\alpha - \beta$  and axis  $\mathbf{h}$ ; it lies inside the body cone.

The above problem is a special case of the motion of a rigid body with a fixed point under no forces, considered in [Chap. XIV](#).

## General motion of a rigid body.

We now consider a rigid body moving quite generally. Let  $\mathbf{F}$  denote the total external force and  $\mathbf{G}$  the total moment of the external forces about the mass center. By (12.203), the acceleration  $\mathbf{f}$  of the mass center (relative to a Newtonian frame) is given by<sup>\*</sup>

$$(12.410) \quad m\mathbf{f} = \mathbf{F},$$

where  $m$  is the mass of the body. For the motion relative to the mass center we have, by (12.209),

$$(12.411) \quad \dot{\mathbf{h}} = \mathbf{G},$$

where  $\mathbf{h}$  is the angular momentum about the mass center. This last equation is exactly the same as if the mass center were fixed, and so can be treated by the methods given above.

Let us resolve the vectors  $\mathbf{f}$ ,  $\mathbf{F}$ ,  $\dot{\mathbf{h}}$ ,  $\mathbf{G}$  along a principal triad  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  at the mass center. As before, this triad is supposed to be permanently a principal triad. Its angular velocity will be denoted by  $\Omega$ ; if the triad is fixed in the body,  $\Omega = \omega$ , the angular velocity of the body. Now, by (12.306),

type, we use the methods of [Chap. VIII](#); similar methods for motion in three dimensions will now be developed.

### General equations of impulsive motion.

Integration of the equations (12.203) and (12.209) from time  $t_0$  to time  $t_1$  gives\*

$$(12.501) \quad \Delta(m\mathbf{q}) = \int_{t_0}^{t_1} \mathbf{F} dt, \quad \Delta\mathbf{h} = \int_{t_0}^{t_1} \mathbf{G} dt,$$

where  $\Delta$  denotes an increment in the time interval  $t_1 - t_0$ . These equations express the principles of linear and angular momentum in integrated form. In words, they read as follows:

(i) the increment in the linear momentum of a system is equal to the *total impulse* of the external forces;

(ii) the increment in the angular momentum about a point  $O$  (either a fixed point in a Newtonian frame or the mass center of the system in question) is equal to the time integral of the total moment about  $O$  of the external forces, i.e., the *total angular impulse* about  $O$ .

In this form the principles of linear and angular momentum can easily be applied to problems where sudden changes in velocity occur. The method of procedure is essentially that given in [Chap. VIII](#), and so we shall give only a brief outline here.

The very short time interval  $t_1 - t_0$  in which the changes occur is regarded as infinitesimal. Any finite force will then contribute nothing to the *total impulsive force*

$$\hat{\mathbf{F}} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \mathbf{F} dt.$$

On the other hand, a force  $\mathbf{P}$ , for which

$$\hat{\mathbf{P}} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \mathbf{P} dt$$

is finite, contributes the *impulsive force*  $\hat{\mathbf{P}}$  to  $\hat{\mathbf{F}}$ . If  $\mathbf{r}$  denotes the position vector of the point of application of such a force  $\mathbf{P}$  (the position vector being relative to the point  $O$  about which the angular momentum is calculated), then  $\mathbf{r}$  does not change by a finite amount in the infinitesimal time  $t_1 - t_0$ . Hence,



$$\lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} (\mathbf{r} \times \mathbf{P}) dt = \mathbf{r} \times \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \mathbf{P} dt = \mathbf{r} \times \hat{\mathbf{P}},$$

and the force  $\mathbf{P}$  contributes the *impulsive moment*  $\mathbf{r} \times \hat{\mathbf{P}}$  to the *total impulsive moment*

$$\hat{\mathbf{G}} = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \mathbf{G} dt.$$

Then, from (12.501), we have

$$(12.502) \quad \Delta(m\mathbf{q}) = \hat{\mathbf{F}}, \quad \Delta\mathbf{h} = \hat{\mathbf{G}},$$

where  $\hat{\mathbf{F}}$  = total impulsive force = vector sum of external impulsive forces,

$\hat{\mathbf{G}}$  = total impulsive moment = total moment of external impulsive forces.

These are the general equations of impulsive motion.

For a rigid body in general motion, the first of the above equations gives the change in the velocity of the mass center; the second gives the change in the angular momentum (and hence the change in angular velocity) about the mass center. For a rigid body with a fixed point, the second equation in (12.502) alone suffices to determine the change in angular velocity.

*Example.* A square plate, of mass  $m$  and edge  $2a$ , is suspended from one corner  $O$ . It is struck at a corner in a horizontal direction perpendicular to the plane of the plate. About what line does the plate begin to turn?

Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  be the principal triad of inertia at  $O$  (Fig. 131);  $\mathbf{i}$  points upward along the diagonal through  $O$ , and  $\mathbf{j}$  is a horizontal vector in the plane of the plate.

Before the plate is struck, the angular momentum about  $O$  is zero; immediately afterward, it is

$$\mathbf{h} = A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k},$$

where  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  are the components of angular velocity and  $A$ ,  $B$ ,  $C$  the principal moments of inertia at  $O$ . If  $\hat{P}$  is the magnitude of the blow, the external impulsive forces are  $\hat{P}\mathbf{k}$  at the point

$$\mathbf{r} = -a\sqrt{2}(\mathbf{i} + \mathbf{j}),$$

and an impulsive reaction  $\hat{\mathbf{Q}}$  at  $O$ . Since  $\hat{\mathbf{Q}}$  has no moment about  $O$ , the second of the equations (12.502) gives

$$A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k} = -a\sqrt{2}(\mathbf{i} + \mathbf{j}) \times \hat{P}\mathbf{k} \\ = a\sqrt{2} \cdot \hat{P}(\mathbf{j} - \mathbf{i}).$$

Hence,

$$\omega_1 = -\frac{a\sqrt{2} \cdot \hat{P}}{A}, \quad \omega_2 = \frac{a\sqrt{2} \cdot \hat{P}}{B}, \quad \omega_3 = 0;$$

the plate begins to turn about a line in its plane passing through  $O$ . The angle  $\theta$ , between  $\mathbf{i}$  and this axis of rotation, is given by

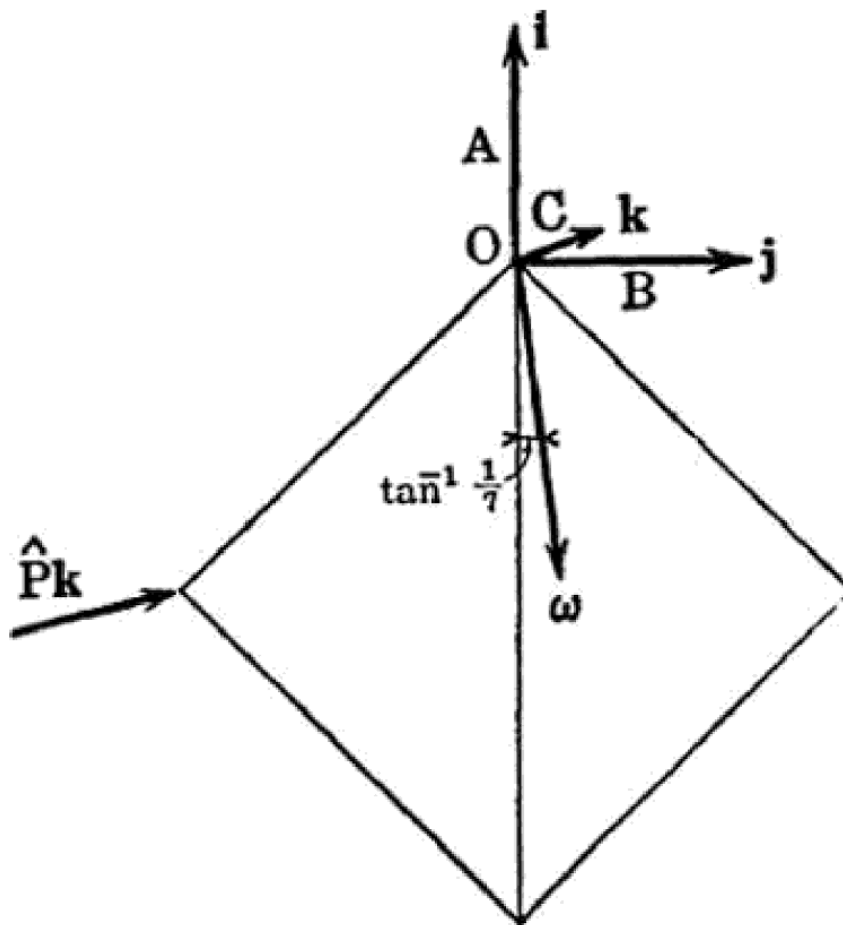


FIG. 131.—Square plate, suspended from  $O$  and struck by a blow  $\hat{P}\mathbf{k}$ .

$$\tan \theta = \frac{\omega_2}{\omega_1} = -\frac{A}{B}.$$

Since

$$A = \frac{1}{3}ma^2, \\ B = \frac{1}{3}ma^2 + 2ma^2 = \frac{7}{3}ma^2,$$

we find

$$\tan \theta = -\frac{1}{7};$$

the axis of rotation is indicated in Fig. 131 by the vector  $\omega$ .

The impulsive reaction  $\hat{\mathbf{Q}}$  at  $O$  is easily found. The velocity of the mass center, immediately after the plate has been hit, is

$$\mathbf{q} = \omega \times (-a\sqrt{2}\mathbf{i}) = \frac{2a^2\hat{P}}{B}\mathbf{k}.$$

Thus, from the first of (12.502), we get

$$\frac{2ma^2\hat{P}}{B}\mathbf{k} = \hat{P}\mathbf{k} + \hat{\mathbf{Q}},$$

and so

$$\hat{\mathbf{Q}} = -\frac{1}{7}\hat{P}\mathbf{k}.$$

## 12.6. SUMMARY OF METHODS OF DYNAMICS IN SPACE

### I. Motion of a particle.

(a) Equations of motion:

$$(12.601) \quad m\mathbf{f} = \mathbf{P} \quad (\text{vector form});$$

$$(12.602) \quad m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z \quad (\text{Cartesian coordinates});$$

$$(12.603) \quad m\dot{s} = P_1, \quad m\frac{\dot{s}^2}{\rho} = P_2, \quad 0 = P_3 \quad (\text{intrinsic equations}).$$

(b) Principle of angular momentum:

$$(12.604) \quad \dot{\mathbf{h}} = \mathbf{r} \times \mathbf{P}.$$

(c) Principle of energy:

$$(12.605) \quad \dot{T} = \dot{W}, \quad (T = \frac{1}{2}mq^2, \quad W = \text{work done});$$

$$(12.606) \quad T + V = E \quad (\text{conservation of energy}).$$

### II. Motion of a system.



(a) Principle of linear momentum:

$$(12.607) \quad \dot{\mathbf{M}} = \mathbf{F}, \quad \left( \mathbf{M} = \sum_{i=1}^n m_i \mathbf{q}_i \right);$$

$$(12.608) \quad m\dot{\mathbf{q}} = \mathbf{F} \quad (\text{motion of mass center}).$$

(b) Principle of angular momentum:

$$(12.609) \quad \dot{\mathbf{h}} = \mathbf{G} \quad (\text{fixed point or mass center}).$$

(c) Principle of energy:

$$(12.610) \quad \dot{T} = \dot{W};$$

$$(12.611) \quad T + V = E \quad (\text{conservation of energy}).$$

### III. Motion of a rigid body.

(a) Rigid body with a fixed point:

$$(12.612) \quad \dot{\mathbf{h}} = \frac{\delta \mathbf{h}}{\delta t} + \boldsymbol{\Omega} \times \mathbf{h} = \mathbf{G};$$

$$(12.613) \quad \begin{cases} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = G_1, \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = G_2, \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = G_3; \end{cases}$$

$$(12.614) \quad \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + V = E \quad (\text{conservative forces}).$$

(b) Rigid body in general:

$$(12.615) \quad \begin{cases} m\mathbf{f} = \mathbf{F} & (\text{motion of mass center}), \\ \dot{\mathbf{h}} = \mathbf{G} & (\text{motion relative to mass center}); \end{cases}$$

$$(12.616) \quad \frac{1}{2}m\dot{\mathbf{q}}^2 + \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + V = E \quad (\text{conservative forces}).$$

### IV. Rotating frame of reference.

Rate of change of any vector:

$$(12.617) \quad \frac{d\mathbf{P}}{dt} = \frac{\delta \mathbf{P}}{\delta t} + \boldsymbol{\Omega} \times \mathbf{P}.$$

### V. Impulsive motion.

General equations:

$$(12.618) \quad \Delta(m\mathbf{q}) = \hat{\mathbf{F}}, \quad \Delta\mathbf{h} = \hat{\mathbf{G}}.$$

## EXERCISES XII

1. A heavy particle moves on a smooth surface. Show that its speed is the same whenever its path cuts a given horizontal curve on the surface.

2. Show directly from Euler's equations (12.404) that, if  $\mathbf{G} = \mathbf{0}$  and  $A = B$ , then  $\omega$  is constant.

3. A particle is attracted toward a fixed line by a force, perpendicular to the line and varying as the distance from the line. Show that its path is a curve traced on an elliptical cylinder.

4. A solid of revolution rotates with constant angular velocity  $\omega$  about a fixed axis which passes through its mass center and is inclined to the axis of symmetry at an angle  $\alpha$ . Prove that the reactions of the axis on the solid are equipollent to a couple of magnitude

$$\pm (C - A)\omega^2 \sin \alpha \cos \alpha,$$

where  $C$  is the moment of inertia about the axis of symmetry and  $A$  the other principal moment of inertia at the mass center.

5. Explain how a man, standing on a smooth sheet of ice, can turn round by moving his arms.

6. A bar of length  $2a$  is fitted at its middle point with a nut which moves without friction on a fixed vertical screw of pitch  $p$ ; the bar remains horizontal and turns with the nut. Find the acceleration of the nut.

7. Two particles, of masses  $m, m'$ , attract one another according to the inverse square law. At time  $t = 0$ ,  $m$  is at the origin and has a velocity  $u$  along the  $x$ -axis, and  $m'$  is at the point  $(a, b, c)$  and has velocity components  $(u', v', w')$ . Determine

- (i) the coordinates of the mass center at time  $t$ ,
- (ii) the constant areal velocity of the motion of  $m'$  relative to  $m$ ,
- (iii) the constant areal velocity of the motion of  $m$  relative to  $m'$ .

8. Two men support a uniform pole of mass  $m$  and length  $2a$  in a horizontal position. They wish to change ends without changing their positions on the ground, by throwing the pole into the air and catching it. If the pole is to remain horizontal throughout its flight and the magnitude of the impulsive force applied by each man is to be a minimum, find the magnitudes and directions of the impulsive forces.

9. An equilateral triangle is formed of three rods, each of mass  $m$  and length  $2a$ . It hangs from one vertex, about which it is free to turn. A blow  $\mathbf{P}$  is struck on one of the lower vertices in a direction perpendicular to the plane of the triangle. Prove that the impulsive reaction on the point of support has a magnitude  $\frac{1}{2}\mathbf{P}$ .

10. A solid homogeneous ellipsoid of mass  $m$  and semiaxes  $a, b, c$  spins with constant angular velocity  $\omega$  about an axis which is fixed in space and makes constant angles  $\alpha, \beta, \gamma$  with the axes of the ellipsoid. Show that the components (along the axes of the ellipsoid) of the couple that must act on it in order to maintain this motion are

$$\frac{1}{2}m\omega^2(b^2 - c^2) \cos \beta \cos \gamma$$

and two similar expressions.

11. A particle of mass  $m$  moves in a plane under the action of two forces. One force is an attraction  $mk^2r$  toward the origin; the other is perpendicular to the velocity  $\mathbf{q}$  and has magnitude  $mk'q$ . Show that the motion is given by an equation of the form

$$x + iy = e^{ik't}(Ae^{ict} + Be^{-ict}).$$

How many arbitrary constants (to fit initial conditions) are present in this solution?

12. An insect runs with constant relative speed  $v$  round the rim of a wheel of radius  $a$  which rolls along a straight road with uniform velocity  $V$ . Find the magnitude and direction of (i) the acceleration relative to the wheel, (ii) the acceleration of transport, and (iii) the Coriolis acceleration. Indicate these accelerations in a diagram.

13. A free rigid body is at rest. Find three linear scalar equations to determine the components of an impulsive force which, applied at an assigned point of the body, imparts to that point an assigned velocity. Solve these equations in the case where the assigned point lies on one of the principal axes of inertia at the mass center.

14. A thin rod of mass  $m$  and length  $2a$  is made to rotate with constant angular velocity  $\omega$  about an axis which passes through one end of the rod and cuts it at a constant angle  $\alpha$ . Reduce the force system exerted by the axis on the rod to a force at the fixed end and a couple.

15. A rigid triangular target is fixed at the corners, and a bullet is fired normally into it. Find the region in which the bullet must strike in order that no support may experience an impulsive reaction normal to the target greater than half the momentum of the bullet.

16. A uniform circular disk of mass  $M$  and radius  $a$  is so mounted that it can turn freely about its center, which is fixed. It is spinning with angular velocity  $\omega$  about the perpendicular to its plane at the center, the plane being horizontal. A particle of mass  $m$ , falling vertically, hits the disk near the edge and adheres to it. Prove that immediately afterward the particle is moving in a direction inclined to the horizontal at an angle  $\alpha$ , given by

$$\tan \alpha = 4 \frac{m(M + 2m)}{M(M + 4m)} \cdot \frac{v}{a\omega},$$

where  $v$  is the speed of the particle just before impact.

17. A homogeneous ellipsoid of semiaxes

$$a, b, c, (a > b > c)$$

is to be mounted on a horizontal axis  $L$  in such a way that it may oscillate as a compound pendulum with the smallest possible periodic time. What position of  $L$  relative to the axes of the ellipsoid should be selected?

**18.** A crankshaft of mass  $m$ , in the form of a letter S formed out of two semicircles, each of radius  $a$ , spins with angular velocity  $\omega$  in bearings at its ends. Find the magnitudes of the reactions exerted on the bearings, and show the directions of these reactions in a diagram.

**19.** A system is set in motion by impulsive forces applied to certain prescribed particles. If  $\hat{\mathbf{P}}$  is the external impulsive force on a typical particle,  $\mathbf{q}$  its velocity, and  $\delta\mathbf{r}$  an arbitrary infinitesimal displacement consistent with the constraints (assumed workless), show that

$$\Sigma(m\mathbf{q} \cdot \delta\mathbf{r}) = \Sigma(\hat{\mathbf{P}} \cdot \delta\mathbf{r}),$$

where the summation on the left extends over all particles of the system and the summation on the right over the prescribed particles.

Let  $T$  be the kinetic energy of the actual motion and  $T'$  that of any other motion ( $\mathbf{q}'$ ) consistent with the constraints and making  $\mathbf{q}' = \mathbf{q}$  for the prescribed particles; prove that  $T < T'$  (Kelvin's theorem).

**20.** A rhombus  $ABCD$  is formed of four uniform rods, each of mass  $m$  and length  $2a$ , smoothly jointed at the vertices. Prove that if the rhombus is in motion in its plane, in such a way that  $A$  and  $C$  are moving along the diagonal  $AC$ , the kinetic energy may be expressed in the form

$$T = 2m(v - 2a\omega \sin \theta)^2 + \frac{8}{3}ma^2\omega^2,$$

where  $v$  is the velocity of  $A$ ,  $\omega$  the angular velocity of  $AB$ , and  $\theta$  the inclination of  $AC$  to  $AB$ .

Hence, prove by Kelvin's theorem (see Exercise 19) that, if the rhombus is at rest in the form of a square and is jerked into motion by an impulsive force applied at  $A$  in the direction  $AC$ , then the angular velocity imparted to the rods is

$$\frac{3\sqrt{2}}{10} \cdot \frac{v}{a},$$

where  $v$  is the velocity imparted to  $A$ .

\* For simplicity, we drop the subscripts from  $\mathbf{q}_0$  and  $\mathbf{f}_0$ , the velocity and acceleration of the mass center.

\* For simplicity, we drop the subscript from  $\mathbf{q}_0$ , the velocity of the mass center.

# CHAPTER XIII

## APPLICATIONS IN DYNAMICS IN SPACE— MOTION OF A PARTICLE

### 13.1. NOTE ON JACOBIAN ELLIPTIC FUNCTIONS

So far, we have used only the elementary functions, alone or in combination—polynomial, trigonometrical, exponential, and logarithmic. We now find it necessary to introduce the *elliptic function*.

#### **Definition of a function by means of a differential equation.**

A variable  $y$  is said to be a function of  $x$  when to values of  $x$  there correspond values of  $y$ . In fact, a function is determined by a *rule* which assigns  $y$  when  $x$  is given. Usually this rule is a formula admitting direct calculation of  $y$  (e.g.,  $x^2$ ,  $3 \sin 2x$ ), but we may also use a differential equation to define a function; we must, however, assign initial conditions to make the solution unique.

Consider, for example, the differential equation

$$\frac{d^2y}{dx^2} + y = 0,$$

with the conditions  $y = 0$ ,  $dy/dx = 1$  for  $x = 0$ . If we had never previously heard of the function  $\sin x$ , this equation and the initial conditions would serve to define it. Another way of defining  $\sin x$  is by the differential equation

$$(13.101) \quad \left(\frac{dy}{dx}\right)^2 = 1 - y^2,$$

with the conditions

$$(13.102) \quad y = 0, \quad \frac{dy}{dx} > 0, \quad \text{for } x = 0.$$

## A type of differential equation with periodic solutions.

In connection with elliptic functions, we have to study the differential equation

$$(13.103) \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2),$$

where  $k$  is a constant such that  $0 < k < 1$ . It is really simpler, however, to take a more general point of view and study first the differential equation

$$(13.104) \quad \left(\frac{dy}{dx}\right)^2 = f(y),$$

where  $f(y)$  is a general function. We can find out a surprising amount about the solutions of this equation without specifying the function  $f(y)$ . We shall, however, assume that  $f(y)$  is continuous; that it vanishes for  $y = a$  and  $y = b$  ( $a < b$ ), but its derivative does not vanish for either of these values; and finally that  $f(y) > 0$  for  $a < y < b$ . (The right-hand side of (13.103) has these properties, if we take  $a = -1$ ,  $b = 1$ .)

Let us take a geometrical point of view, regarding  $x$  and  $y$  as rectangular Cartesian coordinates in a plane. The equation (13.104) is then a relation between the slope and the ordinate on a curve, and a solution, or integral curve, is a curve for which this relation is satisfied.

A number of statements can be made regarding the integral curves of (13.104). These will now be given, followed by their proofs.

(A) If we know an integral curve, then that curve translated through any distance parallel to the  $x$ -axis is also an integral curve.

(B) If an integral curve starts in the fundamental strip  $a \leq y \leq b$ , it cannot pass out of that strip.

(C) Every integral curve in the fundamental strip touches the bounding lines  $y = a$ ,  $y = b$  and has at no other point a tangent parallel to the  $x$ -axis.

(D) There is one, and only one, integral curve touching a bounding line at a given point.\*

(E) All integral curves may be obtained from one integral curve by translation parallel to the  $x$ -axis.

(F) An integral curve is symmetric with respect to its normal at a point of contact with a bounding line.

(G) The  $x$ -distance between successive contacts of an integral curve with

the bounding lines is

$$(13.105) \quad P = \int_a^b \frac{dy}{\sqrt{f(y)}}.$$

(H) Any solution of (13.104),  $y = \phi(x)$ , is a periodic function with period  $2P$ , where  $P$  is given by (13.105); this means that

$$(13.106) \quad \phi(x + 2P) = \phi(x),$$

for all values of  $x$ .

(I) If  $y = \phi(x)$  is any one solution of (13.104), then the general solution is  $y = \phi(x + c)$ , where  $c$  is an arbitrary constant.

Some of these properties are shown in [Fig. 132](#).

#### Proofs:

(A) Neither slope nor ordinate is changed by the translation; if the relation (13.104) is satisfied by the curve before translation, it will be satisfied after translation.

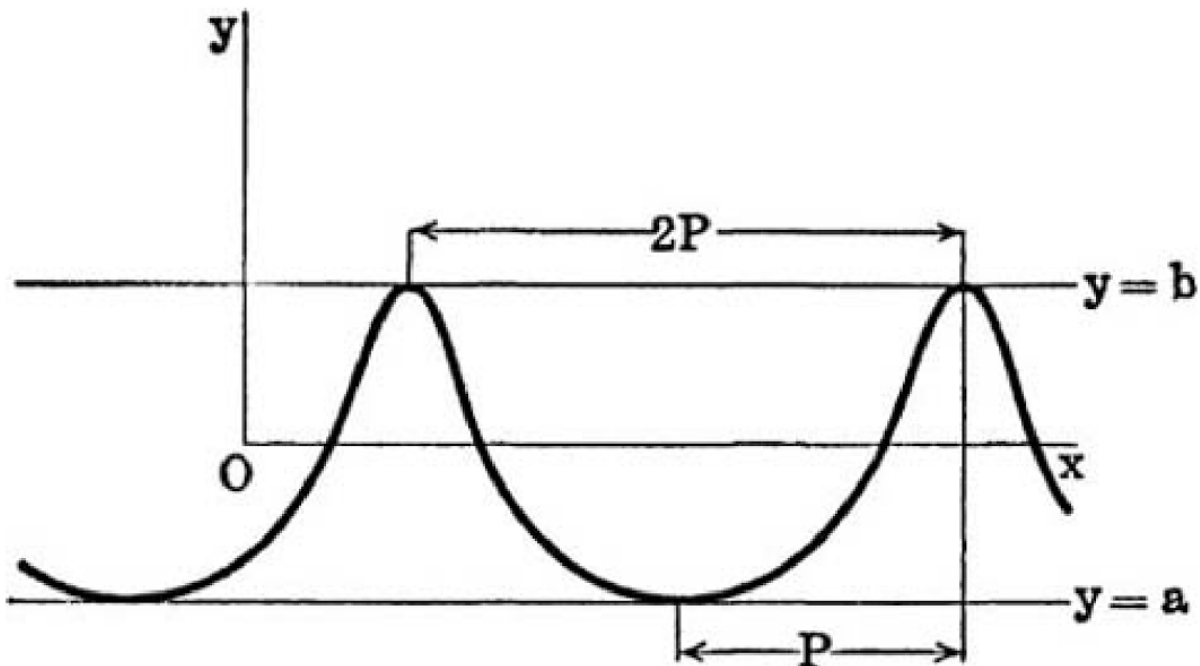


FIG. 132.—General character of a solution of the differential equation (13.104).

(B) If the curve passed out of the strip,  $f(y)$  would become negative and  $dy/dx$  imaginary.

(C) By hypothesis,  $f(a) = f(b) = 0$ ; hence,  $dy/dx = 0$  on the bounding lines. Further,  $f(y) > 0$  for  $a < y < b$ , and so  $dy/dx$  cannot vanish between the bounding lines.



(D) Let  $x = x_0$ ,  $y = a$  be a point on a bounding line. If we invert (13.104), take the square root, and integrate, we get

$$(13.107) \quad x - x_0 = - \int_a^y \frac{d\eta}{\sqrt{f(\eta)}}, \quad x - x_0 = \int_a^y \frac{d\eta}{\sqrt{f(\eta)}},$$

according as  $dy/dx < 0$  or  $dy/dx > 0$ . These two equations together give (in the neighborhood of  $x = x_0$ ) the unique integral curve satisfying the condition of tangency.

(E) Let  $C$  and  $C'$  be any two integral curves. We have to show that  $C'$  may be made to coincide with  $C$  by a translation. Let  $C$  touch  $y = a$  at  $x = x_0$ . Translate  $C'$  until it also touches  $y = a$  at  $x = x_0$ . By (A), it is still an integral curve after translation; by (D), it coincides with  $C$ .

(F) The two equations (13.107) give the two parts of an integral curve, meeting at a point of tangency with  $y = a$ . To a given  $y$ , there correspond equal values of  $x - x_0$ , except for sign. This establishes the symmetry for the parts of the curve running up from  $y = a$  to  $y = b$ . But there is the same symmetry with respect to the normal at a contact with  $y = b$ . It is not difficult to see that this implies symmetry of the whole curve with respect to the normal at any point of contact with a bounding line. If the part of the curve to the right of such a normal is folded over the normal, it will coincide with the part of the curve on the left.

(G) This is obvious from (13.107).

(H) This follows from (G) and the symmetry of the curve.

(I) This merely expresses (E) in analytic form. Since the differential equation is of the first order, we expect just one constant of integration.

## The Jacobian elliptic functions.

Let us now apply our general results to the differential equation

$$(13.108) \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2), \quad (0 < k < 1).$$

The fundamental strip is  $-1 \leq y \leq 1$ . We define the *Jacobian elliptic function*  $\text{sn } x$  to be that solution of (13.108) which satisfies the conditions

$$(13.109) \quad y = 0, \quad \frac{dy}{dx} > 0, \quad \text{for } x = 0.$$

It is evident that  $\text{sn } x$  depends on the value of  $k$ , which is called the *modulus* of

the function, and we may write it  $\text{sn}(x, k)$ ; but it is usual to suppress the explicit dependence on  $k$ . (In speaking of the function, we call it “ess-en-ex.”).

From the result (I), stated on page 366, we know that the most general solution of (13.108) is

$$y = \text{sn}(x + c),$$

where  $c$  is an arbitrary constant. Further, by (H), we know that  $\text{sn } x$  is a periodic function. Thus,

$$(13.110) \quad \text{sn}(x + 4K) = \text{sn } x,$$

where  $K$  is the *elliptic integral*

$$(13.111) \quad K = \frac{1}{2} \int_{-1}^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} \\ = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}};$$

$K$  is, of course, a function of  $k$ .

By (F) the graph of the function  $\text{sn } x$  has symmetry with respect to each normal at a contact with the lines  $y = \pm 1$ . But, since the right-hand side of (13.108) is an even function of  $y$ , the graph has a further symmetry. The equation (13.108) and the conditions (13.109) are unchanged when we change  $x$  into  $-x$  and  $y$  into  $-y$ , and so the curve is unaltered by a reflection in the origin. For the general case, shown in [Fig. 132](#), the whole curve can be constructed by means of the symmetry when we know a half wave, running from  $y = a$  to  $y = b$ . In the case of  $\text{sn } x$ , we need merely know the curve from  $y = 0$  to  $y = 1$  or, equivalently, from  $x = 0$  to  $x = K$ .

The properties of  $\text{sn } x$  may be summed up as follows:

$$(13.112) \quad \begin{cases} \left( \frac{d}{dx} \text{sn } x \right)^2 = (1 - \text{sn}^2 x)(1 - k^2 \text{sn}^2 x), \\ \text{sn } 0 = 0, \quad \left( \frac{d}{dx} \text{sn } x \right)_{x=0} = 1, \\ \text{sn}(x + 4K) = \text{sn } x. \end{cases} \quad (0 < k < 1),$$

We now define other elliptic functions,  $\text{cn } x$  and  $\text{dn } x$ , by the equations

$$(13.113) \quad \begin{cases} \text{cn}^2 x = 1 - \text{sn}^2 x, & \text{cn } 0 = 1, \\ \text{dn}^2 x = 1 - k^2 \text{sn}^2 x, & \text{dn } 0 = 1, \end{cases}$$

The theory of elliptic functions is extensive, but this very brief presentation contains enough to enable us to solve certain dynamical problems. For numerical tables of the functions  $\operatorname{sn} x$ ,  $\operatorname{cn} x$ ,  $\operatorname{dn} x$ , see L. M. Milne-Thomson, *Die elliptischen Funktionen von Jacobi* (Verlag Julius Springer, Berlin, 1931). Tables of elliptic integrals may also be used to find the elliptic functions; cf. J. B. Dale, *Five Figure Tables of Mathematical Functions* (Edward Arnold, London, 1903), or E. Jahnke and F. Emde, *Tables of Functions* (B. G. Teubner, Leipzig, 1938). [Figure 133](#) shows graphs of the functions, drawn for  $k^2 = 0.7$ ; this makes  $K = 2.07536$ .

*Exercise.* Show that, in the limit  $k = 1$ , we get

$$\operatorname{sn} x = \tanh x, \operatorname{cn} x = \operatorname{dn} x = \operatorname{sech} x.$$

## 13.2. THE SIMPLE PENDULUM

### The motion in terms of elliptic functions.

We can now give the exact solution for the motion of a simple pendulum in terms of elliptic functions.\* Let  $m$  be the mass of the bob and  $a$  the length of the pendulum. The equation of energy (12.111) gives

$$(13.201) \quad \frac{1}{2}ma^2\dot{\theta}^2 - mga \cos \theta = E,$$

where  $\theta$  is the inclination of the string to the downward vertical, and  $E$  the constant total energy. We shall suppose the motion to be oscillatory with amplitude  $\alpha$ , so that  $\dot{\theta} = 0$  for  $\theta = \pm\alpha$ . Then  $E = -mga \cos \alpha$ , and (13.201) may be written

$$(13.202) \quad \dot{\theta}^2 = 2p^2 (\cos \theta - \cos \alpha) = 4p^2 (\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\theta),$$

where  $p^2 = g/a$ .

Let us define  $\phi$  by

$$(13.203) \quad \sin \frac{1}{2}\theta = \sin \frac{1}{2}\alpha \sin \phi,$$

so that

$$\frac{1}{2} \cos \frac{1}{2}\theta \dot{\theta} = \sin \frac{1}{2}\alpha \cos \phi \dot{\phi}.$$

Multiplying (13.202) by  $\frac{1}{4}\cos^2 \frac{1}{2}\theta$ , we get

$$\sin^2 \frac{1}{2}\alpha \cos^2 \phi \dot{\phi}^2 = p^2 \sin^2 \frac{1}{2}\alpha \cos^2 \phi \cos^2 \frac{1}{2}\theta,$$

### 13.3. THE SPHERICAL PENDULUM

A particle of mass  $m$  is attached to a fixed point by a light string or rod of length  $a$  and oscillates under the action of gravity. Since the particle is thus constrained to move on a sphere, this system is called a *spherical pendulum*. Under special initial conditions, a spherical pendulum will move in a vertical plane; then the motion is that of a simple pendulum, discussed in Sec. 13.2.

Although we shall be able to determine the general motion of a spherical pendulum in terms of elliptic functions, there are two particular motions which can be discussed quite simply. The first is a motion in which the particle performs small oscillations near the lowest point of the sphere, and the second is motion in a horizontal circle.

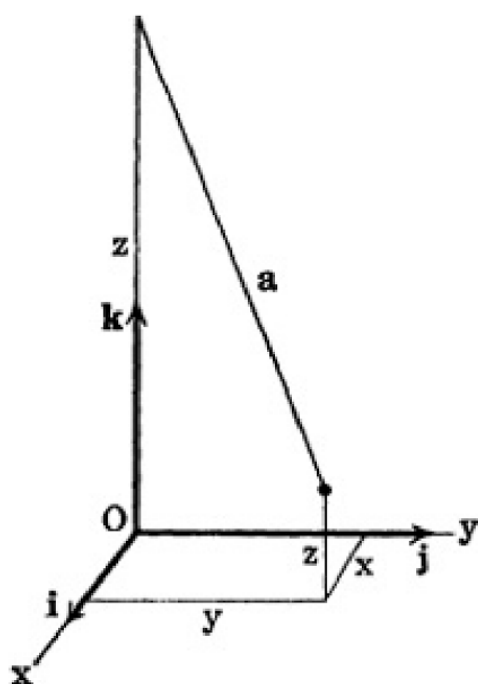


FIG. 134.—Spherical pendulum

#### Small oscillations (first approximation).

Let  $Oxyz$  be rectangular axes,  $O$  being at the lowest point of the sphere and  $Oz$  being directed vertically upward (Fig. 134). If  $i, j, k$  is a unit orthogonal triad along the axes, the position vector of the particle is

$$(13.301) \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The force  $\mathbf{P}$  on the particle is made up of gravity and the tension ( $S$ ) in the string. Now the direction cosines of the string (running from the particle to the point of support) are

$$-\frac{x}{a}, \quad -\frac{y}{a}, \quad \frac{a-z}{a},$$

and so

$$(13.302) \quad \mathbf{P} = -mg\mathbf{k} - [x\mathbf{i} + y\mathbf{j} + (z - a)\mathbf{k}] \frac{S}{a}.$$

So far the expressions are exact. But if  $x$  and  $y$  are small,  $z$  is a small quantity of the second order, since the plane  $z = 0$  touches the sphere. Hence, we have as equation of motion, omitting small quantities of the second order,

$$(13.303) \quad m(x\mathbf{i} + y\mathbf{j}) = -\frac{xS}{a}\mathbf{i} - \frac{yS}{a}\mathbf{j} + (S - mg)\mathbf{k}.$$

Comparing the coefficients, we see that  $S = mg$ , and

$$(13.304) \quad x + p^2x = 0, \quad y + p^2y = 0, \quad \left(p^2 = \frac{g}{a}\right).$$

These are simple harmonic equations, as in (6.403). As far as its projection on the horizontal plane is concerned, the bob of the pendulum moves like a particle attracted toward  $O$  by a force proportional to the distance from  $O$ . As shown in Sec. 6.4, the path is an ellipse with center at  $O$ . (When the ellipse degenerates to a straight line, we get the motion of a simple pendulum, performing small oscillations.)

We have idealized the problem by leaving out all consideration of frictional resistance. The effect of this is to cause the bob of the pendulum to spiral in toward  $O$ , instead of continuing for ever in the elliptical path. However, the approximation (neglect of  $z$ ) is perhaps a more serious oversimplification. We shall see the effect of this later, when we consider the second approximation.

### The conical pendulum.

Any prescribed motion of a particle will take place under the action of a suitable force, namely, a force equal to the acceleration multiplied by the mass of the particle. Thus the bob of a spherical pendulum may be made to move in any way on the sphere defined by the length of the pendulum; to produce this motion, it is in general necessary to add a suitable force to the weight of the bob and the tension in the string. But if we can find a motion in which no such additional force is required, then that motion is a possible motion of the pendulum under weight and tension alone.

Consider a motion in a horizontal circle of radius  $R$  at constant speed  $q$ . The acceleration is of constant magnitude  $q^2/R$  and is directed in along the radius of the circular path. Resolving along this radius, along the tangent to the circular

path, and vertically, we find that no additional force is required provided that

$$S \sin \theta = \frac{mq^2}{R}, \quad S \cos \theta = mg,$$

where  $m$  is the mass of the bob,  $S$  the tension, and  $\theta$  the inclination of the string to the downward vertical (Fig. 135). Since  $\sin \theta = R/a$ , elimination of  $S$  gives

$$(13.305) \quad q^2 = \frac{gR^2}{\sqrt{a^2 - R^2}}.$$

If  $b$  denotes the depth of the horizontal circle below the center of the sphere, so that  $b^2 = a^2 - R^2$ ,

we have

$$(13.306) \quad q^2 = \frac{g(a^2 - b^2)}{b}.$$

This gives the speed  $q$  at which a horizontal circle at depth  $b$  may be described by the bob of the pendulum. When behaving in this way, the pendulum is called a *conical pendulum*, since the string describes a right circular cone.

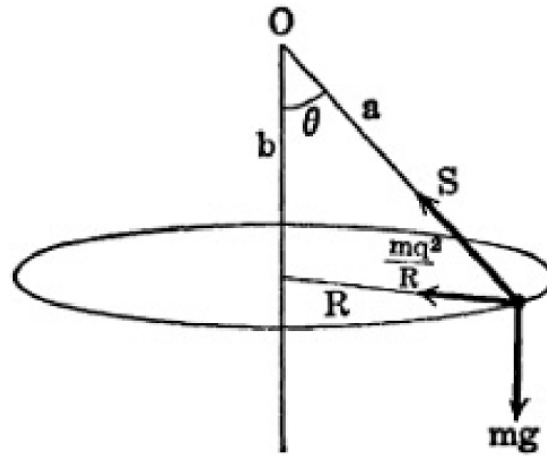


FIG. 135.—Conical pendulum.

*Exercise.* Find the tension in the string of a conical pendulum moving at a depth  $b$ . Examine the limits  $b \rightarrow a$ ,  $b \rightarrow 0$ .

### The general motion of a spherical pendulum.

To investigate the general motion of a spherical pendulum, we take cylindrical coordinates  $R, \phi, z$ , the origin  $O$  being at the center of the sphere and the axis of  $z$  directed vertically upward. We have already obtained the equations of motion in (12.114) and (12.115); they may be written



To obtain  $z$  as a function of  $t$ , we proceed as follows: Since  $z_1, z_2, z_3$  are the zeros of  $f(z)$ , we have

$$(13.311) \quad f(z) = \frac{2g}{a^2} (z - z_1)(z - z_2)(z - z_3).$$

Let us define  $u = \sqrt{z - z_1}$ , so that  $\dot{z} = 2u\dot{u}$ . Then (13.308) gives

$$(13.312) \quad \dot{u}^2 = \frac{g}{2a^2} (z_2 - z_1 - u^2)(z_3 - z_1 - u^2).$$

This suggests the differential equation (13.108) for the elliptic function  $\text{sn}$ . Let us define

$$(13.313) \quad \begin{cases} v = \frac{u}{\sqrt{z_2 - z_1}} = \sqrt{\frac{z - z_1}{z_2 - z_1}}, \\ k = \sqrt{\frac{z_2 - z_1}{z_3 - z_1}}, \quad p = \sqrt{\frac{g(z_3 - z_1)}{2a^2}}. \end{cases}$$

Then (13.312) may be written

$$(13.314) \quad \dot{v}^2 = p^2(1 - v^2)(1 - k^2v^2),$$

and so

$$(13.315) \quad v = \text{sn}[p(t - t_0)],$$

where  $t_0$  is a constant of integration. Hence we have

$$(13.316) \quad \begin{cases} z - z_1 = (z_2 - z_1) \text{sn}^2[p(t - t_0)], \\ z_2 - z = (z_2 - z_1) \text{cn}^2[p(t - t_0)], \\ z_3 - z = (z_3 - z_1) \text{dn}^2[p(t - t_0)]. \end{cases}$$

Any one of these three equations gives  $z$  as a function of  $t$ . We note that  $z$  has the period

$$(13.317) \quad \tau = \frac{2K}{p},$$

where  $K$  is as in (13.111). Since, by (13.307),  $\phi$  increases steadily throughout the motion, the path of the particle on the sphere is as shown in [Fig. 137](#).

We have already seen that the particle oscillates between the two levels  $z = z_1$  and  $z = z_2$ . We shall now show that the arithmetic mean of these levels lies below the center of the sphere, this statement being equivalent to



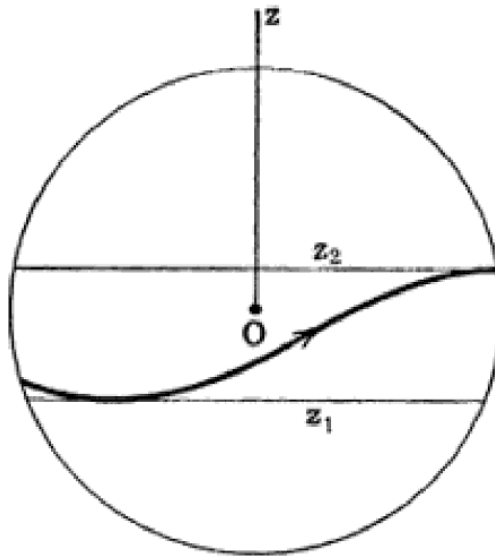


FIG. 137.—The path of the bob of a spherical pendulum.

$$(13.318) \quad z_1 + z_2 < 0.$$

We have two different expressions for  $f(z)$ , (13.309) and (13.311); they must, of course, be identically equal, and therefore

$$(13.319) \quad z_1 + z_2 + z_3 = \frac{E}{g}, \quad z_2 z_3 + z_3 z_1 + z_1 z_2 = -a^2, \\ z_1 z_2 z_3 = \frac{h^2 - 2a^2 E}{2g}.$$

From the second of these, we have

$$(13.320) \quad z_1 + z_2 = -\frac{a^2 + z_1 z_2}{z_3}.$$

Since  $z_1$  and  $z_2$  are each less than  $a$  in absolute value and  $z_3$  is positive, (13.318) follows at once.

The path on the sphere is represented analytically by a relation connecting  $z$  and  $\phi$ . Elimination of  $t$  from (13.307) and (13.308) gives the required relation in the form of a differential equation

$$(13.321) \quad \left(\frac{d\phi}{dz}\right)^2 = \frac{h^2}{(a^2 - z^2)^2 f(z)}.$$

If we look down on the pendulum from a great distance above, the bob appears to describe a plane curve, with  $R$  and  $\phi$  as polar coordinates. The motion resembles that of a particle attracted toward a center of force, the areal velocity ( $\frac{1}{2}R^2\dot{\phi}$ ) being a constant for both motions. Just as we considered the apsides of the orbit of a particle in a plane, so we can consider the apsides of

Substitution in (13.327) gives

$$(13.330) \quad \alpha = \frac{1}{2}\pi \left[ 1 + SD \frac{D^2 - a(z_1 + z_2)}{D^4} \right].$$

It is evident from (13.324) that  $S$  is small; thus the last term is small, and we introduce only a negligible error (of the second order) if we substitute in the fraction

$$D = 2a, z_1 + z_2 = -2a.$$

This gives

$$(13.331) \quad \alpha = \frac{1}{2}\pi \left( 1 + SD \cdot \frac{3}{8a^2} \right).$$

Now if  $R_1, R_2$  are the distances from the central vertical to the apsides, we have accurately

$$R_1^2 = a^2 - z_1^2, \quad R_2^2 = a^2 - z_2^2, \quad SD = R_1 R_2,$$

and so the approximate formula (13.331) becomes

$$(13.332) \quad \alpha = \frac{1}{2}\pi \left( 1 + \frac{3R_1 R_2}{8a^2} \right).$$

We saw, in connection with the equations (13.304), that in the first approximation the path of the particle is a central ellipse. For that curve the apsidal angle is  $\frac{1}{2}\pi$ . Now we see, from (13.332), that the apsidal angle is a little greater than  $\frac{1}{2}\pi$ . This means that the apse advances; the path is approximately a central ellipse, but *this ellipse turns slowly forward* (i.e., in the same sense as that in which the particle describes the path). In one rotation of the particle, the apse advances through an angle

$$(13.333) \quad 4\alpha - 2\pi = \frac{3\pi R_1 R_2}{4a^2} = \frac{3A}{4a^2},$$

where  $A$  is the area of the ellipse. The advance disappears when  $A = 0$ , i.e., when the orbit is flattened into the track of a simple pendulum.

This advance of the apse can be shown by fitting a light writing device to the bob of the pendulum. This traces the rotating elliptical path on a sheet of paper, placed underneath.

#### 13.4. THE MOTION OF A CHARGED PARTICLE IN AN

## ELECTROMAGNETIC FIELD

Much of our knowledge of the structure of matter is derived from the study of the motion of charged particles (electrons or ionized atoms) in electromagnetic fields. Further interest has been added to the problem by the invention of the electron microscope and other devices, in which streams of electrons produce images in much the same way as images are formed by rays of light in an optical instrument.

### Electrostatic and magnetostatic fields.

We shall consider only statical fields, i.e., fields which do not change with time. Such fields are produced by electric charges at rest in condensers or by steady currents; permanent magnets may also be used.

In an electrostatic field, there exists at each point of space an *electric vector*  $\mathbf{E}$ . It is the negative of the gradient of an *electric potential*  $V$ , so that

$$(13.401) \quad \mathbf{E} = - \text{grad } V.$$

The potential  $V$  cannot take arbitrary values throughout space; it must satisfy Laplace's partial differential equation

$$(13.402) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Similarly, in a magnetostatic field there is at each point of space a *magnetic vector*  $\mathbf{H}$ , such that

$$(13.403) \quad \mathbf{H} = - \text{grad } \Omega;$$

$\Omega$  is the *magnetic potential*, and it also satisfies Laplace's equation

$$(13.404) \quad \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{\partial^2 \Omega}{\partial z^2} = 0.$$

Both fields may be present at the same time. The force exerted on a particle carrying an electric charge  $e$ , moving with velocity  $\mathbf{q}$ , is

$$(13.405) \quad \mathbf{P} = e\mathbf{E} + e\mathbf{q} \times \mathbf{H},$$

if the units are suitably chosen.

We accept these basic formulas of electromagnetic theory as the foundation for our dynamical deductions.

Before proceeding to discuss special fields, we shall obtain an equation of energy from (13.405). If  $m$  is the mass of the particle, its equation of motion is\*

$$(13.413) \quad \begin{cases} |\zeta - \left(A - \frac{iFt}{H}\right)| = |B|, \\ \arg \left[ \zeta - \left(A - \frac{iFt}{H}\right) \right] = \arg B - pt. \end{cases}$$

If  $F$  were zero, the first equation would indicate motion in a circle with center  $A$  and radius  $|B|$ , and the second equation would tell us that the circle is described with constant angular velocity  $-p$ . (The sign shows the sense.) The effect of the  $F$ -term is simply to impose an additional motion in which the center of the circle moves with constant complex velocity

$$(13.414) \quad -\frac{iF}{H} = \frac{1}{H} (E_2 - iE_1).$$

This velocity is of magnitude  $E/H$  and is perpendicular to the electric vector.

We sum up our description of the motion of a charged particle in perpendicular uniform electric and magnetic fields as follows: *If started with a velocity perpendicular to  $\mathbf{H}$ , the particle moves as if it were attached to the edge of a circular disk which moves in a plane perpendicular to  $\mathbf{H}$ ; the disk spins with constant angular velocity  $-\epsilon H/m$ , and its center has a constant velocity  $E/H$  perpendicular to  $\mathbf{E}$  (Fig. 138).* The surprising part of this result is that, on the whole, the particle does not move in the direction of the electric field, but perpendicular to it.

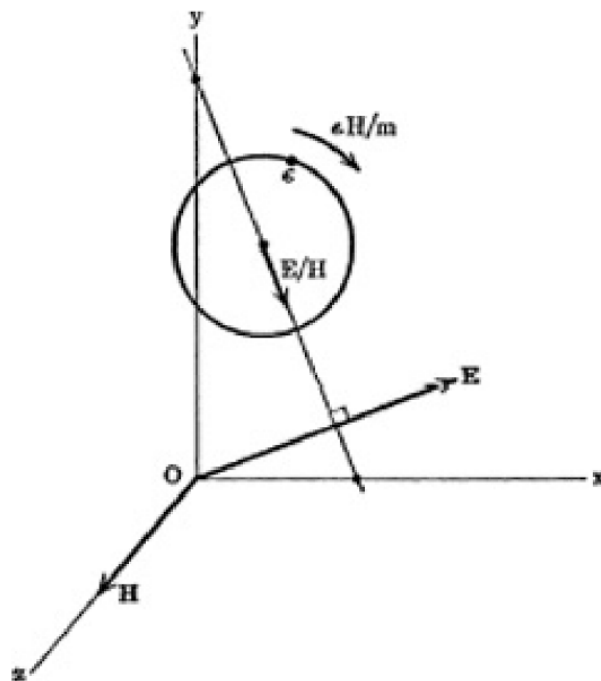


FIG. 138.—Motion of a charged particle in perpendicular uniform electric and magnetic fields.

*Exercise.* Suppose the charged particle starts from rest at the origin at time  $t = 0$ .

Starting from (13.412) prove the following facts concerning the motion:

(i) At time  $t = 2\pi/p$ , it will be at rest again at a distance  $2\pi E/(pH)$  from the origin.

(ii) If  $H$  is very small, and if the particle is allowed to travel for a definite finite time  $t_1$ , then at  $t = t_1$  its complex position is approximately  $\frac{1}{2}\epsilon Ft_1^2/m$ , and its complex velocity is approximately  $\epsilon Ft_1/m$ .

### Motion in a purely electric field and in a purely magnetic field.

We have worked out (13.411) for the general case in which both electric and magnetic fields are present. In the case of a purely electric field ( $H = 0$ ) we return to (13.408), which becomes

$$(13.415) \quad m\ddot{\mathbf{r}} = \epsilon \mathbf{E}.$$

If the field is uniform, the acceleration is constant, and so the particle describes a parabolic trajectory like a projectile under gravity (cf. Sec. 6.1). The plane of the trajectory is determined by the vector  $\mathbf{E}$  and the initial velocity.

In the case of a uniform purely magnetic field ( $E = 0$ ), the equations (13.411) read

$$\zeta = A + Be^{-ip t}, \quad z = C + Dt, \quad \left(p = \frac{\epsilon H}{m}\right).$$

By moving the origin, we can make  $A = C = 0$ ; then we have

$$(13.416) \quad \zeta = Be^{-ip t}, \quad z = Dt.$$

Hence

$$|\zeta| = |B|, \quad \left|\frac{dz}{d\zeta}\right| = \left|\frac{D}{pB}\right|;$$

since these values are constant, it is clear that *the trajectory is a circular helix, with axis parallel to the magnetic field*. The azimuthal angular velocity is  $-p = -\epsilon H/m$ .

The simplest motion in a uniform magnetic field is one in which the initial velocity is perpendicular to the field. Then  $D = 0$  in (13.416), and the trajectory is a circle described with constant speed.

The determination of the charge  $\epsilon$  and the mass  $m$  of an electron is a

problem of great physical interest. Let us see how the results we have established help in that determination. The first thing we notice is that  $\epsilon$  and  $m$  appear in our equations only in the form  $\epsilon/m$ , and therefore it is only this ratio that we can hope to find. It would seem a simple matter to find  $\epsilon/m$  from the circular motion described in the preceding paragraph. The angular velocity is  $-\epsilon H/m$ , and so we have, on equating two different expressions for the angular velocity,

$$\frac{q^2}{R^2} = \frac{\epsilon^2 H^2}{m^2},$$

where  $q$  is the constant speed and  $R$  the radius of the circle. (We have squared the two expressions to avoid the question of sign, which is of no importance here.) If we could measure  $q$ ,  $R$ , and  $H$ , we should at once have  $\epsilon/m$ . Now  $R$  can be measured from a photograph of the track of the electron, and  $H$  can also be measured; but  $q$  presents a difficulty—electrons move too fast for us to find their speeds directly. We have therefore to find  $q$  indirectly. Before entering the magnetic field, the electron is accelerated from rest by an electric field. If it starts from rest at potential  $V = V_0$  and enters the magnetic field with speed  $q$  at potential  $V = V_1$ , then

$$\frac{1}{2}mq^2 = \epsilon(V_0 - V_1)$$

by the principle of energy (13.407). Elimination of  $q$  between the two equations gives

$$\frac{\epsilon}{m} = \frac{2(V_0 - V_1)}{R^2 H^2}.$$

This is a suitable expression for the determination of  $\epsilon/m$ , since all the quantities on the right are measurable. This is the method of Kaufmann. The electromagnetic units are such that (13.405) holds.

### **Axially symmetric fields.**

Let  $R, \phi, z$  be cylindrical coordinates. A field is said to be *axially symmetric* with respect to the  $z$ -axis if the potential is a function of  $R$  and  $z$  only (i.e., independent of  $\phi$ ). An electric field of this type is produced by a system of charged plates perpendicular to the  $z$ -axis, a circular hole with center on the  $z$ -axis being cut from each plate. An axially symmetric magnetic field is produced by currents flowing in circular coils arranged in planes perpendicular

to the  $z$ -axis, the centers of the coils being on the  $z$ -axis.

Let  $V$  be an axially symmetric electric potential. We assume that  $V$  can be expanded in a power series in  $x$  and  $y$ , the coefficients being functions of  $z$ . On account of the axial symmetry,  $x$  and  $y$  can be involved only in the form  $x^2 + y^2$  ( $= R^2$ ), and so the expansion is of the form

$$(13.417) \quad V = V_0(z) + \frac{R^2}{2!} V_1(z) + \frac{R^4}{4!} V_2(z) + \dots$$

Then, by an easy calculation,

$$(13.418) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = [V_0''(z) + 2V_1(z)] \\ + R^2[\frac{1}{2}V_1''(z) + \frac{2}{3}V_2(z)] + \dots$$

In order that Laplace's equation (13.402) may be satisfied, the functions  $V_0, V_1, \dots$  must satisfy the sequence of ordinary differential equations

$$(13.419) \quad V_0''(z) + 2V_1(z) = 0, \quad V_1''(z) + \frac{4}{3}V_2(z) = 0, \dots$$

It is evident that  $V_0(z)$  (the potential on the axis of symmetry) may be chosen arbitrarily, the burden of satisfying Laplace's equations being placed on  $V_1(z), V_2(z), \dots$ .

We may treat an axially symmetric magnetic field in exactly the same way. Assuming for the magnetic potential an expansion of the form

$$(13.420) \quad \Omega = \Omega_0(z) + \frac{R^2}{2!} \Omega_1(z) + \frac{R^4}{4!} \Omega_2(z) + \dots,$$

we deduce the relations

$$(13.421) \quad \Omega_0''(z) + 2\Omega_1(z) = 0, \quad \Omega_1''(z) + \frac{4}{3}\Omega_2(z) = 0, \dots$$

### **Motion of a charged particle near the axis of symmetry of an electromagnetic field.\***

If we introduce the potentials from (13.401) and (13.403), the general equations of motion (13.406) read, when written out in full,



$$(13.422) \quad \begin{cases} \ddot{x} = k \left( \frac{\partial V}{\partial x} + \dot{y} \frac{\partial \Omega}{\partial z} - \dot{z} \frac{\partial \Omega}{\partial y} \right), \\ \ddot{y} = k \left( \frac{\partial V}{\partial y} + \dot{z} \frac{\partial \Omega}{\partial x} - \dot{x} \frac{\partial \Omega}{\partial z} \right), \\ \ddot{z} = k \left( \frac{\partial V}{\partial z} + \dot{x} \frac{\partial \Omega}{\partial y} - \dot{y} \frac{\partial \Omega}{\partial x} \right), \end{cases}$$

where  $k = -e/m$ . In the most interesting applications, the charged particle is an electron carrying a negative charge; in this case  $k$  is positive.

Let us assume that the electromagnetic field has the  $z$ -axis for axis of symmetry, so that we have the expansions (13.417) and (13.420) for  $V$  and  $\Omega$ , respectively. We shall consider only motion near the axis of symmetry, so that  $x$ ,  $y$ , and their derivatives are small. Then, neglecting terms of order higher than the first, we rewrite (13.422) in the form

$$(13.423) \quad \begin{cases} \ddot{x} = k(xV_1 + \dot{y}\Omega'_0 - \dot{z}y\Omega_1), \\ \ddot{y} = k(yV_1 + \dot{z}x\Omega_1 - \dot{x}\Omega'_0), \\ \ddot{z} = kV'_0, \end{cases}$$

where the prime denotes  $d/dz$ . The last of these equations is equivalent, in our approximation, to the equation of energy (13.407), which may be written

$$(13.424) \quad \dot{z}^2 = 2k(V_0 - C),$$

where  $C$  is a constant. This constant may be determined when the initial values of  $z$  and  $\dot{z}$  are given.

We note that (13.424) determines  $\dot{z}$  as a function of  $z$ , to within a sign. Let us assume that  $\dot{z}$  is positive throughout the motion. Then we may write

$$(13.425) \quad \dot{z} = w(z) > 0, \quad w^2 = 2k(V_0 - C).$$

The function  $w$  is the axial component of velocity. Since we are neglecting  $\dot{x}^2$  and  $\dot{y}^2$ , it is clear that, to our order of approximation,  $w$  also represents the magnitude of the velocity vector. By (13.419) and (13.421) we have

We now choose  $v$  so as to make the coefficient of  $u'$  vanish. We do this by writing

$$(13.432) \quad v = \exp \left[ -\frac{1}{2} \int_{z_0}^z P(\xi) d\xi \right],$$

where  $z_0$  is the initial value of  $z$ . When we substitute (13.432) in (13.431) we get, after some easy calculation,

$$(13.433) \quad \begin{cases} u'' + S(z)u = 0, \\ S(z) = Q - \frac{1}{4}P^2 - \frac{1}{2}P'. \end{cases}$$

Also, by (13.430),

$$(13.434) \quad \zeta = u \exp \left[ -\frac{1}{2} \int_{z_0}^z P(\xi) d\xi \right].$$

When we substitute for  $P$  and  $Q$  the expressions given in (13.429), we obtain a much simpler expression for  $S$  than we might expect, and (13.433) reads

$$(13.435) \quad u'' + S(z)u = 0, \quad S(z) = \frac{3}{4} \frac{w'^2}{w^2} + \frac{k^2 \Omega_0'^2}{4w^2}.$$

The relation between the actual displacement vector  $\zeta$  and the artificial displacement vector  $u$  is

$$(13.436) \quad \zeta = u \sqrt{\frac{w_0}{w}} \exp \left[ -\frac{1}{2} i k \int_{z_0}^z \frac{\Omega_0'(\xi)}{w(\xi)} d\xi \right],$$

where  $w_0$  is the value of  $w$  when  $z = z_0$ .

It may be convenient for reference to write the results separately for the cases of electrostatic and magnetostatic fields:

*Electrostatic field:*

$$(13.437) \quad \left\{ \begin{array}{l} u'' + S(z)u = 0, \quad S(z) = \frac{3}{4} \frac{w'^2}{w^2} \\ \qquad \qquad \qquad = \frac{3V_0'^2}{16 \left( V_0 - V_{00} + \frac{w_0^2}{2k} \right)^2}, \\ w(z) = \sqrt{2k[V_0(z) - C]}, \quad k = -\frac{\epsilon}{m}, \\ \qquad \qquad \qquad \zeta = u \sqrt{\frac{w_0}{w}}, \\ V_{00} = V_0(z_0), \quad w_0 = w(z_0). \end{array} \right.$$

*Magnetostatic field:*

$$(13.438) \quad \left\{ \begin{array}{l} u'' + S(z)u = 0, \quad S(z) = \frac{k^2 \Omega_0'^2}{4w_0^2}, \\ w_0 = \text{constant velocity}, \quad k = -\frac{\epsilon}{m}, \\ \qquad \qquad \qquad \zeta = u \exp \left[ \frac{ik}{2w_0} (\Omega_{00} - \Omega_0) \right], \\ \Omega_{00} = \Omega_0(z_0). \end{array} \right.$$

There are some remarkable features in the preceding work. In (13.429) the coefficients  $P$  and  $Q$  were *complex*; but when we transform to (13.435), we get a *real* coefficient  $S$ . Not only is  $S$  real, it is also *positive*; this has an important bearing on the formation of “images” by an axially symmetric electromagnetic field, as we shall see later.

The mathematical difference between the electrostatic case and the magnetostatic case is less than we might expect. In each case the coefficient  $S$  is positive. The chief difference lies in the relation between  $\zeta$  and  $u$ . In the electrostatic case, the connection is real, and the complex vector  $\zeta$  has the same direction as the complex vector  $u$ . In the magnetostatic case the connection is complex; the magnitudes of the two vectors are equal, and  $\zeta$  is obtained from  $u$  by rotation through an angle

$$\frac{k}{2w_0} (\Omega_{00} - \Omega_0).$$

To sum up, in the general electromagnetic case, the determination of the path of the particle involves the solution of (13.435). As initial conditions, we

may assume that the particle starts from the point  $(z_0, \zeta_0)$  with velocity  $w_0$  in a direction giving to  $\zeta'$  the value  $\zeta'_0$ . Thus, (13.435) is to be solved, with the initial conditions for  $z = z_0$ ,

$$(13.439) \quad u = \zeta_0, \quad u' = \zeta'_0 + \frac{1}{2}k\zeta_0 \left( \frac{V'_{00}}{w_0^2} + \frac{i\Omega'_{00}}{w_0} \right),$$

where

$$V'_{00} = V'_0(z_0), \quad \Omega'_{00} = \Omega'_0(z_0).$$

In the electrostatic case, (13.437) replaces (13.435), and we put  $\Omega'_{00} = 0$  in (13.439); in the magnetostatic case, (13.438) replaces (13.435), and we put  $V'_{00} = 0$  in (13.439).

*Exercise.* Show that the small angle between the initial velocity vector and the axis of symmetry is  $|\zeta'_0|$ .

## The electromagnetic lens.

In an optical instrument, such as a microscope, camera, or telescope, rays of light are bent by a system of glass lenses, the system usually having an axis of symmetry. The function of the instrument is to produce an image, the rays from each point of the object being brought to a focus at an image point. In recent times, there has been a remarkable development of electromagnetic devices analogous to the image-forming optical instrument. Instead of rays of light bent by glass lenses, there are streams of electrons whose trajectories are curved by means of electromagnetic fields.\* When an axially symmetric electromagnetic field is used, the equation (13.435) is the fundamental equation from which the trajectories of the electrons are determined.

Let  $Oz$  (Fig. 139) be the axis of symmetry of an electromagnetic field; let  $\Pi_0$  be a plane perpendicular to this axis, with the equation  $z = z_0$ . Let  $P_0$  be any point on  $\Pi_0$  with a complex position vector  $x + iy = \zeta_0$ . We suppose that from the point  $P_0$  there are projected a number of identical charged particles (electrons). Their velocities have a common magnitude  $w_0$ , but their directions are different; the directions are, however, nearly parallel to the axis of symmetry, so that our methods apply.

The trajectory of each electron satisfies (13.435). The function  $w(z)$  is given by (13.425). Since all the electrons have the same charge  $e$ , the same mass  $m$ , and the same initial velocity  $w_0$ , the constant  $C$  has the same value for all the

trajectories. Hence  $w$  [and consequently  $S$  in (13.435)] is the same for all the trajectories. This is, of course, a mathematical idealization. As far as we know, all electrons have the same charge and the same mass, but we cannot secure accurately a common initial velocity. Hence, in practice, the constant  $C$  and the functions  $w$  and  $S$  will not be quite the same for all the trajectories. This leads to what is called, from the optical analogue, “chromatic aberration,” the velocity of the electron corresponding to the color of the light. But for our purposes we shall neglect this effect and regard  $w$  and  $S$  as the same for all the trajectories.

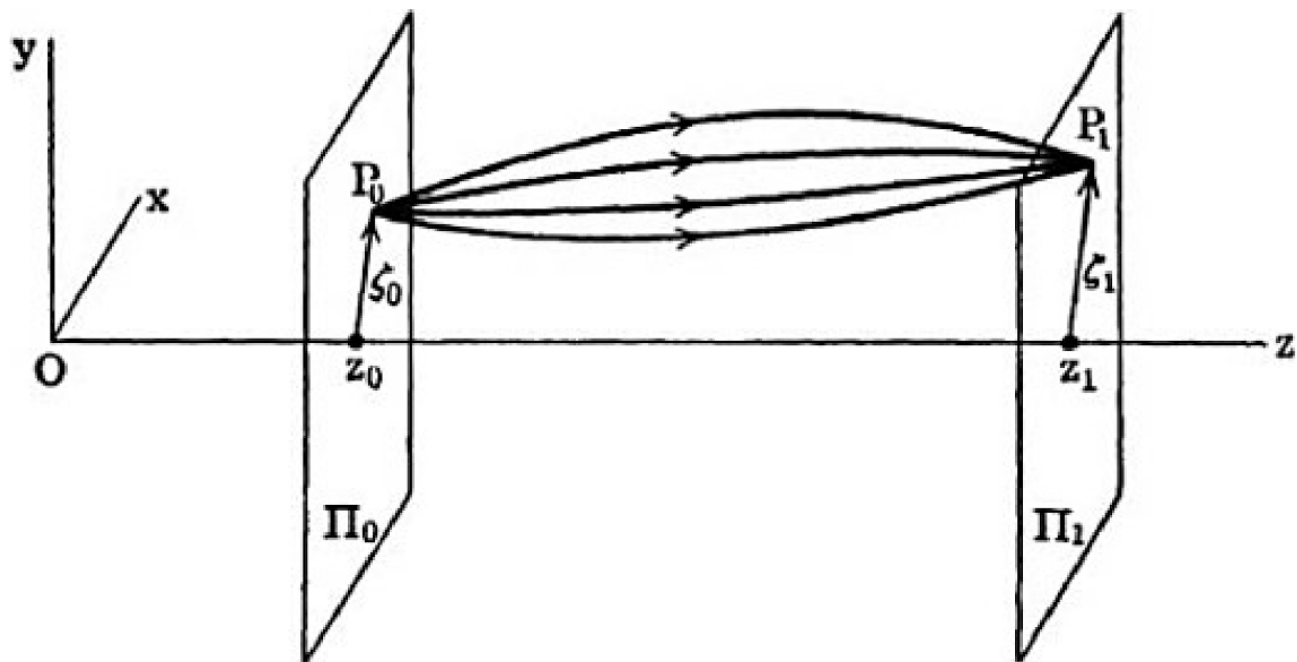


FIG. 139.—Formation of an Image

We note that, from (13.439), for  $z = z_0$  we have  $u = \zeta_0$  for all the electrons; but the initial value of  $u'$  depends on the particular electron since  $\zeta'_0$  is not the same for them all.

It is known from the theory of linear differential equations that the general solution of (13.435) is of the form<sup>\*</sup>

$$(13.440) \quad u = \alpha f(z) + \beta g(z),$$

where  $\alpha, \beta$  are arbitrary constants of integration (which may be complex) and  $f(z), g(z)$  are independent particular solutions. Since the coefficient  $S$  in (13.435) is real, we can obtain two real independent particular solutions by taking the initial conditions

$$(13.441) \quad f(z_0) = 0, \quad f'(z_0) = 1; \quad g(z_0) = 1, \quad g'(z_0) = 0.$$

With this choice of  $f$  and  $g$ , it follows from (13.440) that

$$\alpha = u'_0, \quad \beta = u_0,$$

where  $u_0, u'_0$  are the values of  $u, u'$  when  $z = z_0$ ; then (13.440) may be written

$$(13.442) \quad u = u'_0 f(z) + u_0 g(z).$$

Let  $u_1$  be the value of  $u$  at the point where the trajectory cuts the plane  $z = z_1$ ; then

$$(13.443) \quad u_1 = u'_0 f(z_1) + u_0 g(z_1).$$

In the family of trajectories which we are considering, i.e., a family starting from a point  $(z_0, \zeta_0)$ ,  $u_0$  has a common value but  $u'_0$  changes from trajectory to trajectory. Thus, in general, (13.443) will give an *area* on the plane  $z = z_1$  when we substitute the various values of  $u'_0$  corresponding to the various initial directions. The equation (13.443) will define a *single point* on the plane  $z = z_1$  if, and only if,

$$(13.444) \quad f(z_1) = 0.$$

Let us recall that the function  $f(z)$  is defined by the following differential equation and initial conditions:

$$(13.445) \quad \begin{cases} f''(z) + S(z)f(z) = 0, \\ f(z_0) = 0, \quad f'(z_0) = 1. \end{cases}$$

Can we find a plane  $z = z_1$  (other than  $z = z_0$ ) such that the whole family of trajectories cut it in a single point? This is equivalent to asking whether the equation (13.444) has a solution other than  $z_1 = z_0$ .

Although we cannot give a definite answer to this question in general, we can discuss it qualitatively. Consider the graph of  $f(z)$ . This graph starts from the  $z$ -axis at  $z_0$ , sloping up at  $45^\circ$ . Thus  $f(z)$  is positive at first; hence by (13.445), since  $S$  is positive,  $f''(z)$  is negative, and it remains negative as long as  $f(z)$  is positive. This means that the graph is convex when viewed from above. Either of two things happens. The curve may turn down and cut the  $z$ -axis at some point  $z_1$  (Fig. 140a); or it may turn so slowly that it reaches  $z = \infty$  before coming down to the  $z$ -axis (Fig. 140b). In the former case, the equation (13.444) has a solution; in the latter case, it has no solution (at least not for values of  $z_1$  greater than  $z_0$ , and we are interested only in such values). In



general terms, we may say that the larger  $S$  is, the more chance there is that there will be a solution; because the larger  $S$  is, the more rapidly does the graph of  $f(z)$  turn downward.

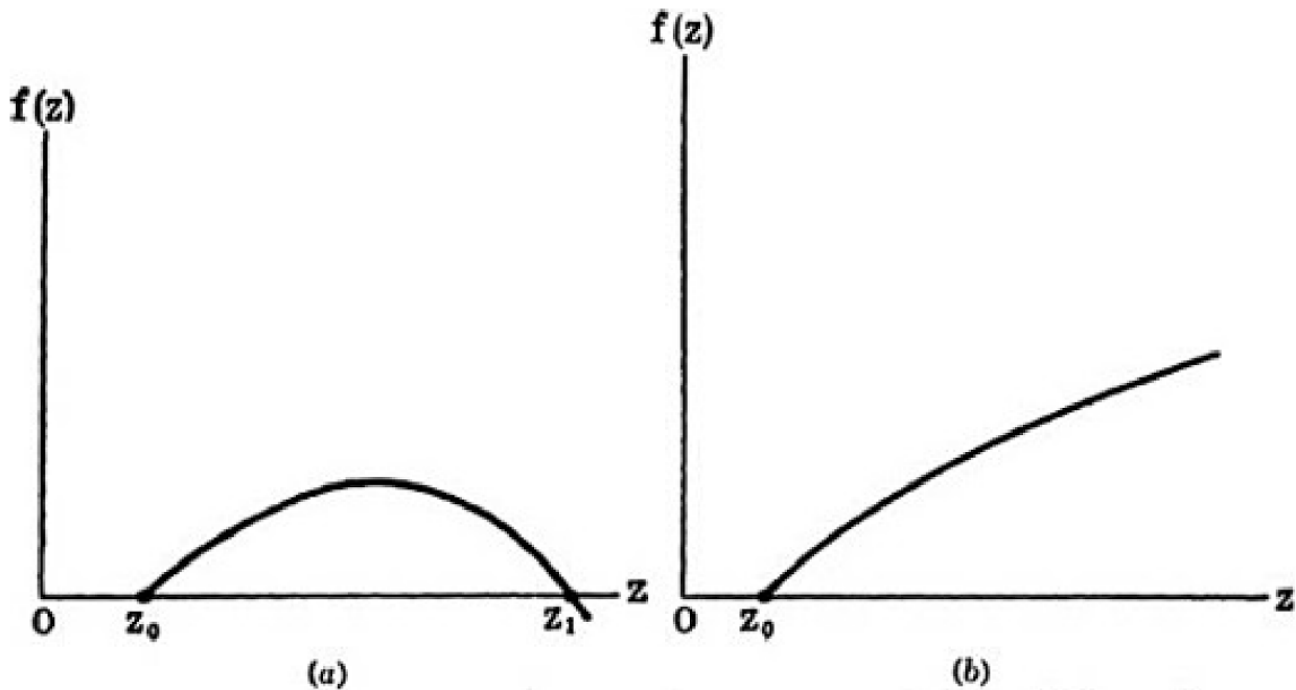


FIG. 140.—Graph of  $f(z)$ : (a) in the case where an image is formed, (b) in the case where an image is not formed.

If (13.444) has a solution, then the family of trajectories starting out from a point  $P_0$  meet again in a point  $P_1$ , as shown in Fig. 139. Borrowing the language of optics, we may call  $P_0$  the *object point* and  $P_1$  the *image point*. In this sense, an *axially symmetric electromagnetic field may form images*. We might go further and say that it *will* form images if it is strong enough, because  $S$  is increased by an increase in the strength of the field.

It will be noted that the equation (13.444), which determines the plane  $\Pi_1$  on which the image is formed, does not involve  $\zeta_0$ . Consequently we may state the following important result: *If an object point  $P_0$  on the plane  $\Pi_0$  has an image  $P_1$  on the plane  $\Pi_1$ , then every object point on the plane  $\Pi_0$  (near the axis of symmetry, to make the approximate method valid) has an image on the plane  $\Pi_1$ .* In fact, we have an *object plane* and an *image plane*, just as in the optics of a lens. Hence, by analogy, we may speak of an axially symmetric electromagnetic field as an *electromagnetic lens*.

Suppose that on the object plane  $\Pi_0$  there is some minute structure which we wish to photograph. We set up a photographic plate at the plane  $\Pi_1$  and bombard the plane  $\Pi_0$  from the left with a stream of electrons. Each point of  $\Pi_0$  becomes a source of electrons travelling on towards  $\Pi_1$  and converging to an



where  $\Omega_{01}$  is the value of  $\Omega_0$  at  $z = z_1$ . The magnification is  $m = |g(z_1)|$ . The image vector  $\zeta_1$  is obtained by first applying this magnification to the object vector  $\zeta_0$  and then rotating it about the  $z$ -axis through an angle

$$(13.452) \quad \alpha = \frac{k}{2w_0} (\Omega_{00} - \Omega_{01}).$$

Figures 141*a* and *b* show the projections of object point  $P_0$ , image point  $P_1$ , and trajectories on the plane  $z = 0$ . They are drawn for  $m = 2$  and  $\alpha = \pi/4$ . In actual electron microscopes the magnification may be as high as 200,000.

### Approximations for electromagnetic lenses.

The determination of the focal properties of an electromagnetic lens depends, as we have seen, on the solution of the differential equation

$$(13.453) \quad f''(z) + S(z)f(z) = 0,$$

with the initial conditions, as in (13.441),

$$(13.454) \quad f(z_0) = 0, \quad f'(z_0) = 1.$$

There is no simple way of solving this equation, and we have to fall back on approximate methods. We shall consider the case where the electromagnetic field is concentrated on a short length of the  $z$ -axis, so that there is practically no field outside this short range. Making a mathematical idealization, we shall assume that there is a field for  $-h < z < h$  and no field outside that range.

In the absence of electric field, the axial potential  $V_0$  is constant, and so, by (13.425),  $w$  is constant and  $w' = 0$ . In the absence of magnetic field, the axial potential  $\Omega_0$  is constant and  $\Omega'_0 = 0$ . Thus, by (13.435),  $S = 0$  for that range of values of  $z$  for which the electromagnetic field vanishes. By hypothesis, this is the case outside the range  $-h < z < h$ , and then the differential equation (13.453) becomes very simple:  $f''(z) = 0$ .

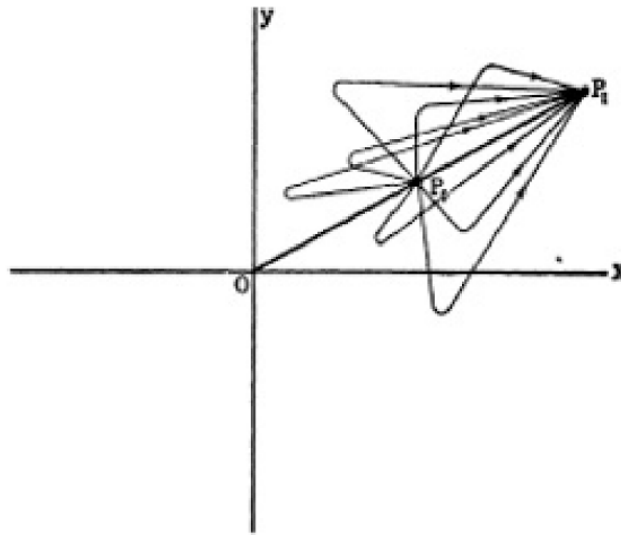


FIG. 141*a*.—Formation of an image by an electrostatic lens.

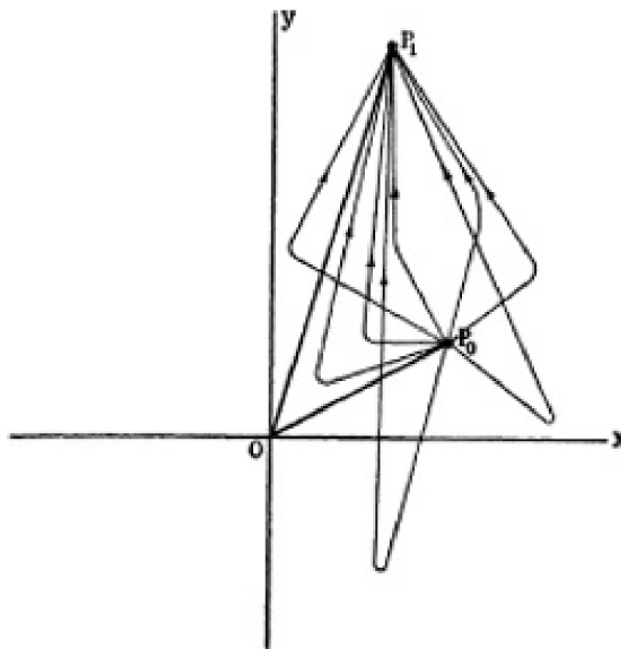


FIG. 141*b*.—Formation of an image by a magnetostatic lens.

The function  $f(z)$  is therefore a linear function of  $z$ . This corresponds to the fact that, in the absence of electromagnetic field, an electron travels in a straight line with constant velocity. The projections of trajectories in Figs. 141*a* and *b* are drawn for such a case; each projection consists of two straight lines, connected by a curve. The curve is produced by the action of a concentrated electromagnetic field. If the field extended

$$(13.458) \quad \begin{cases} f(h) = h - z_0 - \int_{-h}^h d\eta \int_{-h}^{\eta} S(\xi) f(\xi) d\xi, \\ f'(h) = 1 - \int_{-h}^h S(\xi) f(\xi) d\xi. \end{cases}$$

At first sight it may appear that we have found the values of  $f$  and its derivative on leaving the field, but of course this is illusory, because we do not know the function  $f$  occurring in the integrals. However, we can use the above equations as a basis for approximation.

If  $S(z)$  is finite and  $h$  is small, it is evident from (13.458) that the changes in  $f$  and  $f'$  in passing from  $z = -h$  to  $z = h$  are small. But to get an image, the graph of  $f$  must be bent through an angle of more than  $45^\circ$  on passing through the field. In fact, there must be a finite change in  $f'$ , and consequently we must use a strong field. It is clear that  $S(z)$  must be large of the order  $h^{-1}$ . Then the integral in the second of (13.458) is finite; the double integral in the first of (13.458) is small of order  $h$ , showing that although the change in  $f'$  is finite, the change in  $f$  is small.

Before introducing the approximation, let us get an expression for  $z_1$ , the coordinate of the image point. From the linearity of the graph of  $f$  outside the field, we have

$$(13.459) \quad f'(h) = -\frac{f(h)}{z_1 - h},$$

or

$$(13.460) \quad \frac{1}{z_1 - h} = -\frac{f'(h)}{f(h)}.$$

This will give us  $z_1$  if we can evaluate  $f(h)$  and  $f'(h)$  from (13.458).

We now take up the method of approximate solution by the method of iteration. The key equation is (13.457). For  $f$  under the sign of integration, we substitute  $f$  as given by the equation itself. This does not get rid of  $f$  on the right-hand side, but it pushes it under more signs of integration and thus reduces its importance. This procedure gives

$$(13.461) \quad f(z) = z - z_0 - \int_{-h}^z d\eta \int_{-h}^{\eta} S(\xi) d\xi \left[ \xi - z_0 - \int_{-h}^{\xi} dq \int_{-h}^q S(p) f(p) dp \right].$$

The multiple integrals are to be evaluated starting from the right hand side. We can rewrite this in the form

$$(13.462) \quad f(z) = z - z_0 + z_0 \int_{-h}^z d\eta \int_{-h}^{\eta} S(\xi) d\xi \\ - \int_{-h}^z d\eta \int_{-h}^{\eta} \xi S(\xi) d\xi + \int_{-h}^z d\eta \int_{-h}^{\eta} S(\xi) d\xi \int_{-h}^{\xi} dq \int_{-h}^q S(p) f(p) dp.$$

This expression is accurate. For  $z$  in the range  $-h < z < h$ , the first integral is small of order  $h$ , and the remaining integrals are small of order  $h^2$ . Differentiation of (13.462) gives

$$(13.463) \quad f'(z) = 1 + z_0 \int_{-h}^z S(\xi) d\xi - \int_{-h}^z \xi S(\xi) d\xi \\ + \int_{-h}^z S(\xi) d\xi \int_{-h}^{\xi} dq \int_{-h}^q S(p) f(p) dp.$$

Here the first integral is finite, and the remaining integrals small, of order  $h$ . We observe that in (13.462) and (13.463), only the last integrals are unknown.

If we required a higher approximation, we could substitute again for  $f$  under the sign of integration; then the integrals containing  $f$  would be small of order  $h^3$  in the expression for  $f(z)$ , and small of order  $h^2$  in the expression for  $f'(z)$ . This process could be continued indefinitely.

Let us, however, content ourselves with approximations for  $f(h)$  and  $f'(h)$  which retain terms of order  $h$  but reject terms of order  $h^2$ . We shall commit an error only of order  $h^2$  if we substitute  $f(p) = -z_0$  in the last integral in (13.463). Accordingly, putting  $z = h$ , we get the following approximate expressions:

$$(13.464) \quad \begin{cases} f(h) = h - z_0 + z_0 \int_{-h}^h d\eta \int_{-h}^{\eta} S(\xi) d\xi, \\ f'(h) = 1 + z_0 \int_{-h}^h S(\xi) d\xi - \int_{-h}^h \xi S(\xi) d\xi \\ \quad - z_0 \int_{-h}^h S(\xi) d\xi \int_{-h}^{\xi} dq \int_{-h}^q S(p) dp. \end{cases}$$

To write these results more neatly, we introduce the finite constants

$$(13.465) \quad \begin{cases} A = \int_{-h}^h S(\xi) d\xi, & B = h^{-1} \int_{-h}^h \xi S(\xi) d\xi, \\ D = h^{-1} \int_{-h}^h S(\xi) d\xi \int_{-h}^{\xi} (\xi - \eta) S(\eta) d\eta. \end{cases}$$

We note that, by inversion of order of integration,

$$(13.466) \quad \begin{cases} \int_{-h}^h d\eta \int_{-h}^{\eta} S(\xi) d\xi = \int_{-h}^h (h - \xi) S(\xi) d\xi = h(A - B), \\ \int_{-h}^h S(\xi) d\xi \int_{-h}^{\xi} dq \int_{-h}^q S(p) dp \\ \quad = \int_{-h}^h S(\xi) d\xi \int_{-h}^{\xi} (\xi - p) S(p) dp = hD. \end{cases}$$

Consequently, (13.464) read

$$(13.467) \quad \begin{cases} f(h) = -z_0 + h[1 + (A - B)z_0], \\ f'(h) = 1 + z_0A - h(B + Dz_0). \end{cases}$$

The constants  $A$ ,  $B$ ,  $D$  may be evaluated numerically if the field is given; it should be noted however that they involve also the initial velocity  $w_0$ , since  $S$  involves  $w_0$  (cf. equation (13.435)).

*Exercise 1.* Show that if  $S = K/h$ , a constant, for  $-h < z < h$ , then

$$A = 2K, \quad B = 0, \quad D = \frac{1}{2}K^2.$$

*Exercise 2.* Evaluate  $A$ ,  $B$ , and  $D$ , if, for  $-h < z < h$ ,

$$S(z) = ah^{-1} \cos^2 \frac{\pi z}{2h},$$

where  $a$  is a constant.

Using (13.467) in (13.460), the image  $z_1$  corresponding to an object  $z_0$  is given by

$$(13.468) \quad \frac{1}{z_1 - h} = \frac{1 + z_0A - h(B + Dz_0)}{z_0 - h[1 + (A - B)z_0]},$$

or more symmetrically, to the same order of approximation (i.e. neglecting  $h^2$ ),

$$(13.469) \quad \frac{1}{z_1} - \frac{1}{z_0} - A = -h \left( \frac{B}{z_1} + \frac{B}{z_0} + D \right).$$

If we let  $z_0 \rightarrow -\infty$ , we get

$$(13.470) \quad \frac{1}{z_1} - A = -h \left( \frac{B}{z_1} + D \right).$$

The value of  $z_1$  so obtained gives the image of an object at infinity.

If we are satisfied with the rougher approximation in which  $h$  is neglected, (13.469) becomes

$$(13.471) \quad \frac{1}{z_1} - \frac{1}{z_0} = A.$$

If we let  $z_0 \rightarrow -\infty$ , the corresponding value of  $z_1$  is called the *focal length*  $F$  of the electromagnetic lens; by (13.471), we have

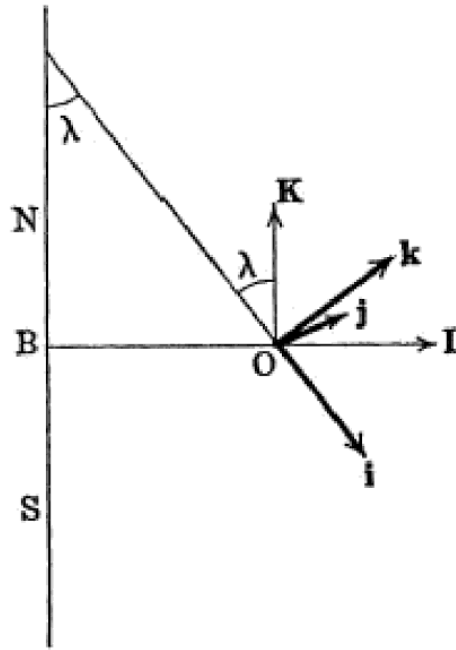


FIG. 143.—Vectors used in discussing the effects of the earth's rotation.

Let us put  $\vec{BO} = \mathbf{a}$  and denote by  $\mathbf{r}$  the position vector of a moving particle relative to  $O$ . Then the position vector of the particle relative to  $B$  is

$$(13.501) \quad \mathbf{r}_B = \mathbf{a} + \mathbf{r}.$$

Since  $B$  is a fixed point in a Newtonian frame of reference, the absolute acceleration of the particle is

$$(13.502) \quad \ddot{\mathbf{r}}_B = \ddot{\mathbf{a}} + \ddot{\mathbf{r}}.$$

Here  $\ddot{\mathbf{a}}$  is the acceleration of  $O$ ; since  $O$  moves in a circle with constant angular velocity  $\Omega$ , we have

$$(13.503) \quad \ddot{\mathbf{a}} = -a\Omega^2\mathbf{I} = -a\Omega^2(\sin \lambda \mathbf{i} + \cos \lambda \mathbf{k}).$$

Let  $m$  be the mass of the particle. The force of gravity is proportional to  $m$  and may be written  $m\mathbf{F}$ . We denote by  $\mathbf{P}$  the resultant of all other forces. The equation of motion is

$$(13.504) \quad m\ddot{\mathbf{r}}_B = m\mathbf{F} + \mathbf{P},$$

or, by (13.502) and (13.503),

$$(13.505) \quad m\ddot{\mathbf{r}} = m\mathbf{F} + \mathbf{P} + ma\Omega^2(\sin \lambda \mathbf{i} + \cos \lambda \mathbf{k}).$$

Let us apply this equation to a plumb line, hanging in equilibrium with the bob at  $O$ . Then  $\mathbf{P}$  is the tension in the plumb line and points in the direction  $\mathbf{k}$ . As in Sec. 5.3, we define  $g$  to be this tension, divided by the mass of the bob, so that

$$\mathbf{P} = mg\mathbf{k}.$$

Since  $\mathbf{r} = \mathbf{0}$ , (13.505) gives

$$(13.506) \quad \mathbf{F}_0 + a\Omega^2(\sin \lambda \mathbf{i} + \cos \lambda \mathbf{k}) = -g\mathbf{k},$$

where  $\mathbf{F}_0$  is the force of gravity per unit mass at  $O$ .

Now  $\mathbf{F}$  in the general equation (13.505) and  $\mathbf{F}_0$  in (13.506) are not equal vectors unless the particle is at  $O$ , for the earth's gravitational field changes from point to point. But we shall assume that the particle always stays so close to  $O$  that variations in the earth's field are negligible. So we introduce our first approximation, putting

$$(13.507) \quad \mathbf{F} = \mathbf{F}_0$$

in the equation of motion (13.505). When, further, we substitute for  $\mathbf{F}_0$  from (13.506), we get for any moving particle

$$(13.508) \quad m\ddot{\mathbf{r}} = \mathbf{P} - mg\mathbf{k}.$$

Our next task is to resolve this equation into components along  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ . This triad turns with a constant angular velocity

$$(13.509) \quad \dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega} \times \mathbf{K} = -\Omega \cos \lambda \mathbf{i} + \Omega \sin \lambda \mathbf{k},$$

and so, by (12.310),

$$(13.510) \quad \ddot{\mathbf{r}} = \frac{\delta^2 \mathbf{r}}{\delta t^2} + 2\boldsymbol{\Omega} \times \frac{\delta \mathbf{r}}{\delta t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}).$$

We now introduce a second approximation, dropping the last term on account of the smallness of  $\Omega$ . Substitution from (13.510) in (13.508) gives the *vector form of the equations of motion relative to the earth's surface*,

$$(13.511) \quad m \frac{\delta^2 \mathbf{r}}{\delta t^2} = \mathbf{P} - mg\mathbf{k} - 2m\boldsymbol{\Omega} \times \frac{\delta \mathbf{r}}{\delta t}.$$

Here  $\delta^2 \mathbf{r} / \delta t^2$  and  $\delta \mathbf{r} / \delta t$  are, respectively, the relative acceleration and velocity; the last term is the Coriolis force. The centrifugal force has been eliminated in two steps, first by (13.506) and secondly by neglect of the last term in (13.510). It will be noticed that (13.511) is essentially the same differential equation as (13.406) or (13.408)—the equation of motion of a charged particle in an electromagnetic field.

Let us now introduce axes  $Oxyz$  coincident in direction with  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ , so that



$$(13.517) \quad \begin{cases} x = u_0 t + \Omega v_0 t^2 \sin \lambda, \\ z = w_0 t - \frac{1}{2} g t^2 + \Omega v_0 t^2 \cos \lambda. \end{cases}$$

Two cases are of particular interest, a particle dropped from rest, and a particle representing a projectile fired with high velocity in a flat trajectory.

In the case of a particle dropped from rest, we put

$$u_0 = v_0 = w_0 = 0,$$

and get

$$(13.518) \quad x = 0, \quad y = \frac{1}{3} \Omega g t^3 \cos \lambda, \quad z = -\frac{1}{2} g t^2.$$

The path is a semicubical parabola in the east-west vertical plane,

$$(13.519) \quad y^2 = -\frac{8}{9} \cdot \frac{\Omega^2 \cos^2 \lambda}{g} \cdot z^3.$$

It is evident from (13.518) that *the deviation from the vertical is toward the east*. From (13.519), the deviation for fall from a height  $h$  is

$$\frac{1}{3} \Omega \cos \lambda \cdot 2h \sqrt{\frac{2h}{g}}.$$

This is zero at the poles ( $\lambda = \pm \frac{1}{2}\pi$ ), as we should expect.

For a projectile with large  $u_0$  and  $v_0$ , we neglect the term in  $w_0$  and also the last term in (13.516). Thus the projection of the trajectory on the horizontal plane has the equations

$$(13.520) \quad x = u_0 t + \Omega v_0 t^2 \sin \lambda, \quad y = v_0 t - \Omega u_0 t^2 \sin \lambda.$$

These may be expressed in complex form in the single equation

$$(13.521) \quad x + iy = (u_0 + iv_0)(t - i\Omega t^2 \sin \lambda).$$

Let us put

$$x + iy = R e^{i\phi}, \quad u_0 + iv_0 = q_0 e^{i\alpha}$$

and write, as we may since  $\Omega$  is small,

$$1 - i\Omega t \sin \lambda = \exp(-i\Omega t \sin \lambda).$$

Then (13.521) takes the form

$$R e^{i\phi} = q_0 t \exp(i\alpha - i\Omega t \sin \lambda),$$

North Pole, the angular velocity becomes— $\Omega$  and may be regarded as due directly to the earth's rotation beneath the pendulum.

When we discussed the spherical pendulum in Sec. 13.3, without taking the earth's rotation into consideration, we started with a first approximation and obtained an elliptical orbit from equations (13.304). We then proceeded to a second approximation and found, in (13.333), an expression for the rate at which the elliptical orbit advances. In the case of Foucault's pendulum, the situation is more involved. In the first place, the angular velocity  $\Omega$  of the earth is small (cf. page 143), and that fact was used in obtaining (13.512). Secondly, we have considered only small oscillations (first approximation). If we were to proceed to a second approximation, we would find two superimposed rotations of the elliptical orbit—one depending, as in (13.333), on the area of the orbit (area effect), and the other an angular velocity— $\Omega \sin \lambda$  due to the earth's rotation (Foucault effect). Unless special precautions are taken, the area effect is likely to be much larger than the Foucault effect and to conceal it. To prevent this, it is usual to draw the pendulum aside with a thread and start the motion by burning the thread. This means that, for  $t = 0$ , we have  $\zeta = 0$  and  $\dot{\zeta} = \dot{\zeta}_0$  (say). Then, by (13.529),

$$(13.531) \quad A + B = \zeta_0, \quad p(A - B) - (A + B)\Omega \sin \lambda = 0.$$

Now (13.530) may be written

$$(13.532) \quad \zeta_1 = (A + B) \cos pt + i(A - B) \sin pt,$$

or, by (13.531),

$$(13.533) \quad \zeta_1 = \zeta_0 (\cos pt + i \sin pt \cdot \Omega \sin \lambda / p).$$

It is easy to see that this represents motion in an ellipse with semiaxes  $|\zeta_0|$  and  $|\zeta_0|\Omega \sin \lambda / p$ . For an ellipse with these semiaxes, described in a Newtonian frame of reference, the advance of the apse in one period ( $2\pi/p$ ) is, by (13.333),

$$(13.534) \quad \frac{3\pi}{4a^2p} |\zeta_0|^2 \Omega \sin \lambda,$$

and so the angular velocity of the ellipse due to the area effect is

$$(13.535) \quad \frac{3|\zeta_0|^2}{8a^2} \Omega \sin \lambda.$$

Since  $|\zeta_0|/a$  is small, this angular velocity is much smaller than the Foucault angular velocity— $\Omega \sin \lambda$  and may be regarded as negligible. Hence, if the pendulum is started by burning a thread, the rotation of the orbit may be

regarded as due to the Foucault effect alone.

It should be noted that, for any small elliptical orbit, there is a distinction between the area effect and the Foucault effect. The area effect is always a rotation in the sense in which the ellipse is described and reverses when that sense is reversed, but the Foucault rotation takes place in a definite sense (clockwise in the Northern Hemisphere and counterclockwise in the Southern).

### 13.6. SUMMARY OF APPLICATIONS IN DYNAMICS IN SPACE— MOTION OF A PARTICLE

#### I. Jacobian elliptic functions.

(a) Differential equation:

$$(13.601) \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2), \quad (0 < k < 1).$$

(b) General solution:

$$(13.602) \quad y = \operatorname{sn} (x + c).$$

(c) Other elliptic functions:

$$(13.603) \quad \operatorname{cn}^2 x = 1 - \operatorname{sn}^2 x, \quad \operatorname{dn}^2 x = 1 - k^2 \operatorname{sn}^2 x;$$

$$(13.604) \quad \frac{d}{dx} \operatorname{sn} x = \operatorname{cn} x \operatorname{dn} x, \quad \frac{d}{dx} \operatorname{cn} x = -\operatorname{sn} x \operatorname{dn} x, \\ \frac{d}{dx} \operatorname{dn} x = -k^2 \operatorname{sn} x \operatorname{cn} x.$$

(d) Periodicity:

$$(13.605) \quad \operatorname{sn} (x + 4K) = \operatorname{sn} x, \quad \operatorname{cn} (x + 4K) = \operatorname{cn} x, \\ \operatorname{dn} (x + 2K) = \operatorname{dn} x;$$

$$(13.606) \quad K = \int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

#### II Simple pendulum.

(a) Motion:

$$(13.607) \quad \begin{cases} \sin \frac{1}{2}\theta = \sin \frac{1}{2}\alpha \operatorname{sn} [p(t - t_0)], \\ p^2 = \frac{g}{a}, \quad k = \sin \frac{1}{2}\alpha. \end{cases}$$

(b) Periodic time:

$$(13.608) \quad \tau = \frac{4}{p} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\ = 2\pi \sqrt{\frac{a}{g}} (1 + \frac{1}{8}k^2), \text{ approximately.}$$

### III. Spherical pendulum.

(a) General motion: the pendulum oscillates between two levels, found by solving a cubic equation; the analytical solution is

$$(13.609) \quad \begin{cases} z - z_1 = (z_2 - z_1) \operatorname{sn}^2 [p(t - t_0)], \\ p^2 = \frac{g}{a}, \quad k^2 = \frac{z_2 - z_1}{z_3 - z_1}. \end{cases}$$

(b) Small oscillations: in the first approximation the path is an ellipse; in the second approximation the ellipse turns at a rate proportional to its area.

### IV. Motion of a charged particle in an electromagnetic field.

(a) Uniform electric field: the trajectory is a parabola.

(b) Uniform magnetic field: the trajectory is a helix.

(c) Axially symmetric electromagnetic field: the trajectory satisfies

$$(13.610) \quad \begin{cases} u'' + S(z)u = 0, \\ S(z) = \frac{3}{4} \frac{w'^2}{w^2} + \frac{k^2 \Omega_0'^2}{4w^2} = \frac{3V_0'^2}{16 \left( V_0 - V_{00} + \frac{w_0^2}{2k} \right)^2} \\ \quad + \frac{k \Omega_0'^2}{8 \left( V_0 - V_{00} + \frac{w_0^2}{2k} \right)}, \\ \zeta = x + iy = u \sqrt{\frac{w_0}{w}} \exp \left( -\frac{1}{2}ik \int_{z_0}^z \frac{\Omega_0'(\xi)}{w(\xi)} d\xi \right), \\ w = \text{speed of particle} = \sqrt{2k \left( V_0 - V_{00} + \frac{w_0^2}{2k} \right)}, \\ w_0 = w(z_0), \quad k = -\frac{e}{m}, \\ V_0 = \text{axial electric potential}, \quad V_{00} = V_0(z_0), \\ \Omega_0 = \text{axial magnetic potential}, \\ \text{Prime indicates } d/dz. \end{cases}$$

(d) Electromagnetic lens:

(i) Image plane  $z = z_1$  of object plane  $z = z_0$  determined by

$$(13.611) \quad \begin{cases} f(z_1) = 0, \\ f''(z) + S(z)f(z) = 0, \\ f(z_0) = 0, \quad f'(z_0) = 1; \end{cases}$$

(ii) Magnification given by

$$(13.612) \quad \begin{cases} m = |g(z_1)| \sqrt{\frac{w_0}{w_1}}, \\ g''(z) + S(z)g(z) = 0, \\ g(z_0) = 1, \quad g'(z_0) = 0, \\ w_1^2 = w_0^2 + 2k[V_0(z_1) - V_0(z_0)]. \end{cases}$$

(e) Magnetostatic lens: rotation of image given by

$$(13.613) \quad \alpha = \frac{k}{2w_0} [\Omega_0(z_0) - \Omega_0(z_1)].$$

## V. Effects of the earth's rotation.

(a) Equations of motion:

$$(13.614) \quad \begin{cases} m\ddot{x} = X + 2m\Omega \sin \lambda \cdot \dot{y}, \\ m\ddot{y} = Y - 2m\Omega(\sin \lambda \cdot \dot{x} + \cos \lambda \cdot \dot{z}), \\ m\ddot{z} = Z - mg + 2m\Omega \cos \lambda \cdot \dot{y}, \\ X, Y, Z = \text{force other than gravity}, \\ \lambda = \text{latitude}. \end{cases}$$

(b) A falling body deviates to the east.

(c) A projectile deviates to the right in the Northern Hemisphere.

(d) Foucault's pendulum turns with angular velocity— $\Omega \sin \lambda$ , clockwise in the Northern Hemisphere.

## EXERCISES XIII

1. A simple pendulum of mass  $m$  and length  $a$  performs finite oscillations, the greatest inclination of the string to the vertical being  $30^\circ$ . Find the tension in the string when the bob is in its highest position.

2. The string of a spherical pendulum is held out horizontally and the bob started with a horizontal velocity perpendicular to the string. Find to the nearest foot per second the magnitude of this velocity if the minimum inclination of

the string to the vertical in the subsequent motion is  $45^\circ$ . The length of the string is 54 inches.

3. A particle carrying a charge  $e$  is projected from the origin with a velocity  $u_0$  in the direction of the  $x$ -axis. There is a uniform magnetic field of strength  $H$  parallel to the  $z$ -axis. If the particle crosses the plane  $x = 0$  at a distance  $a$  from the origin, find its mass. (Isotopes are separated in a mass spectroscopy by a method such as this.)

4. A heavy bead is free to move on a smooth circular wire of radius  $a$ , which rotates with constant angular velocity  $\Omega$  about a fixed vertical diameter. Find the possible positions of relative equilibrium. If  $\Omega^2 > g/a$ , find the period of small oscillations about a position of stable equilibrium.

5. A stone is thrown straight up, rises to a height of 100 ft., and falls to earth. Estimate the deviation due to the rotation of earth, the latitude of the place being  $45^\circ$  North. (Neglect air resistance.)

6. A particle moves under gravity on a smooth surface of revolution with axis vertical. The equation of the surface in cylindrical coordinates is given in the form  $R = F(z)$ . If the velocity is horizontal and of magnitude  $q_1$  at a height  $z_1$ , and again horizontal and of magnitude  $q_2$  at a height  $z_2$ , determine  $q_1$  and  $q_2$  in terms of  $z_1$  and  $z_2$ . (Use the principles of energy and angular momentum.)

7. In a simple pendulum the bob is connected by a light string of length  $a$  to the fixed point of support. The bob starts in the lowest position with speed  $q_0$ . Show that if

$$2ga < q_0^2 < 5ga,$$

the string will slacken during the motion, so that the bob falls inward from the circular path.

8. Show that, on account of the rotation of the earth, a train traveling south exerts a slight sideways force on the western rail of the track. Give an approximate expression for this force in terms of the mass of the train, its speed, the latitude, and the angular velocity of the earth.

9. A particle moves on a smooth surface of revolution with axis vertical. The equation of the surface in cylindrical coordinates is  $R = F(z)$ . Use the principles of angular momentum and energy to show that

$$R^2 \dot{\phi} = h, \\ \frac{1}{2}(z^2 + \dot{R}^2 + R^2 \dot{\phi}^2) + gz = E,$$

where  $h$  and  $E$  are constants. Deduce that  $z$  satisfies a differential equation of the form

that the motion of a conical pendulum is stable.

**18.** A particle moves under gravity on a rough vertical circle. It starts from rest at one end of the horizontal diameter and comes to rest at the lowest point of the circle. Find an equation to determine the coefficient of friction.

**19.** A heavy bead starts from rest at a point  $A$  and slides down a smooth wire in the form of a helix having the parametric equations

$$x = a \cos \theta, y = a \sin \theta, z = a\theta \tan \alpha,$$

the axis of  $z$  being vertical. When it is vertically under  $A$ , a second bead starts from rest at  $A$ . Show that the tangential acceleration of each bead is  $g \sin \alpha$ , and deduce a formula determining all the subsequent instants at which one bead is vertically underneath the other.

**20.** A heavy particle is constrained to move on the inner surface of a smooth right circular cone of semivertical angle  $\alpha$ , the axis of the cone being vertical and the vertex down. The particle is in steady motion in a circle at height  $b$  above the vertex. Find the periodic time for small oscillations about this steady motion.

**21.** A particle slides in a smooth straight tube which rotates with constant angular velocity  $\omega$  about a vertical axis which does not intersect the tube. The tube is inclined at an angle  $\alpha$  to the vertical. Initially the particle is projected upward along the tube with speed  $q_0$ , relative to the tube, from the point where the shortest distance between the axis and the tube meets the tube. Show that, no matter what the length of the tube may be, the particle will escape at the upper end provided

Image

**22.** Consider an axially symmetric electric field in which the axial potential is of the form  $V_0 = az + b$ . Show that the field is uniform throughout space and parallel to the  $z$ -axis. Verify directly from (13.429) that the trajectory of a charged particle is parabolic.

Consider also the case of an axially symmetric magnetic field with the axial potential of the form  $\Omega_0 = az + b$ . Verify from (13.438) that the trajectory is a helix.

**23.** A particle moves under gravity on a smooth surface, the principal radii of curvature at its lowest point being  $a, b$  ( $a > b$ ). The surface rotates with constant angular velocity  $\omega$  about the normal at the lowest point. Show that, if  $Oxy$  are horizontal axes attached to the surface at the lowest point and directed along the lines of curvature at that point, then the equation of motion for small



vibrations near the lowest point are

Image

Considering solutions of the form  $x = Ae^{nt}$ ,  $y = Be^{nt}$ , deduce that there will be instability if  $\omega^2$  lies between  $g/a$  and  $g/b$ .

**24.** If in a magnetostatic lens the axial component  $H$  of the magnetic vector is constant, show that an object point on the axis at  $z = 0$  will give an image at

Image

where  $w_0$  is the initial velocity.

**25.** For a magnetostatic lens, with the field concentrated in  $-h < z < h$ , prove that, to the order  $h$  inclusive, the magnification of an object in the plane  $z = z_0$  is

$$|1 + Az_0 - h(B + Dz_0)|^{-1},$$

where  $A, B, D$  are constants defined by (13.465) Evaluate explicitly, if the axial component of magnetic field  $H$  is constant in  $-h < z < h$ .

\* The bounding lines  $y = a$ ,  $y = b$  satisfy (13.104), but we do not regard them as integral curves. They are singular solutions.

\* Since the motion of a compound pendulum is identical with that of the equivalent simple pendulum (cf. Sec. 7.2), the solution now given applies also to the compound pendulum; in (13.202) and the subsequent equations, we are to put  $p^2 = ga/k^2$ , where  $a$  is the distance of the mass center from the axis of suspension and  $k$  the radius of gyration about that axis.

\* This is the nonrelativistic equation of motion and is a good approximation if the velocity of the particle is small compared with the velocity of light. In the accurate relativistic equation, we replace the left-hand side of (13.406) by Image; see Chap. XVI.

\* Reference may be made to N. Chako and A. A. Blank, Supplementary Note No. I, in R. K. Luneberg, Mathematical Theory of Optics (Brown University, 1944)

\* Cf. L. M. Myers, Electron Optics (Chapman & Hall, Ltd., London, 1939), p. 100.

\* Cf. E. L. Ince, Ordinary Differential Equations (Longmans, Green & Co., Ltd., London, 1927), p. 119.

## CHAPTER XIV

### APPLICATIONS IN DYNAMICS IN SPACE— MOTION OF A RIGID BODY

#### 14.1. MOTION OF A RIGID BODY WITH A FIXED POINT UNDER NO FORCES

If a rigid body is constrained to turn about a smooth fixed axis, under no forces other than the reaction of the axis, the motion is extremely simple: the body spins with constant angular velocity. But if, instead of fixing a *line* in the body, we fix one *point* only, the motion under no forces is much more complicated. The problem of finding this motion is of wider interest than might appear at first sight, for the motion of a rigid body relative to its mass center is the same as if the mass center were fixed (cf. Sec. 12.4).

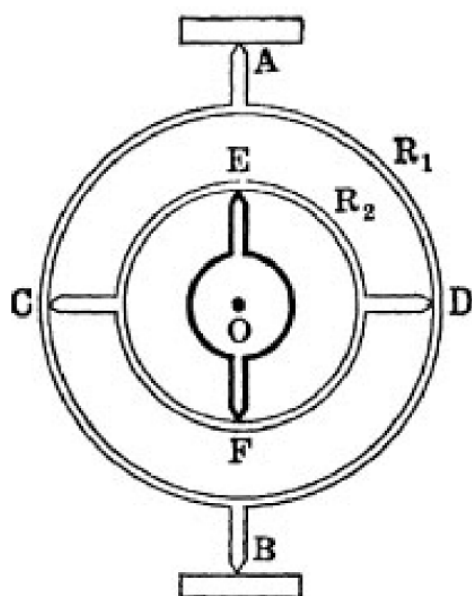


FIG. 144.—Cardan's suspension.

The mounting of a body so as to fix only one point is much more complicated than that required to give it a fixed line. It may be done by an arrangement of light rings, known as “Cardan's suspension” (Fig. 144). The body is represented by the inner circle. The points  $A$ ,  $B$  are fixed. Rotation of

the ring  $R_1$  about  $AB$  gives one degree of freedom. Rotation of the ring  $R_2$  about  $CD$  gives a second degree of freedom. Rotation of the body itself about  $EF$  gives the third. The body can take up all positions in which the point  $O$  of the body is fixed in space,  $O$  being the common intersection of  $AB$ ,  $CD$ , and  $EF$ . All the apparatus, except the body itself, is to be regarded as massless in the mathematical theory; this cannot, of course, be achieved in practice, but the masses of  $R_1$  and  $R_2$  are made as small as possible compared with the mass of the body.

There are two ways of treating the problem of the motion of a body with a fixed point under no forces—the descriptive and the analytic. The descriptive method, or *method of Poinsot*, gives a good qualitative idea of the motion. In the case where the body has an axis of dynamical symmetry, the description is particularly simple; we shall consider that case in detail later.

### The method of Poinsot.

Let  $O$  be the fixed point in the body, and  $A, B, C$  the principal moments of inertia at  $O$ . Let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be unit vectors fixed in the body and directed along the principal axes at  $O$ . For the angular velocity and angular momentum we have, by (11.509),

$$(14.101) \quad \boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}, \quad \mathbf{h} = A\omega_1 \mathbf{i} + B\omega_2 \mathbf{j} + C\omega_3 \mathbf{k}.$$

When we say that the body is under no forces, we mean more precisely that the forces acting on the body have no moment about  $O$ . (Thus our argument applies to a heavy body under the action of gravity, provided that  $O$  is the center of gravity.) Since the external forces do no work and have no moment about  $O$ , we have the following facts to assist us in discussing the motion:

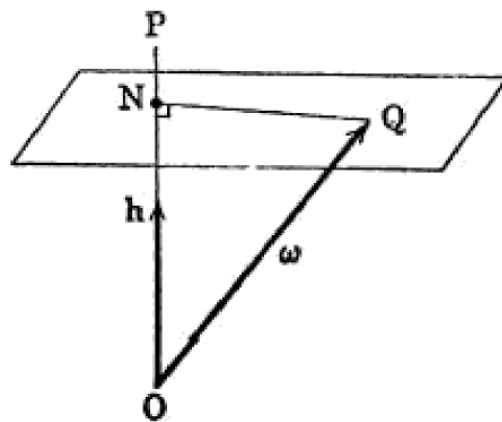


FIG. 145.—The invariable line and the invariable plane.

- (i) the kinetic energy  $T$  is constant;
- (ii) the angular momentum  $\mathbf{h}$  is a constant vector.

From the first of these we have, by (11.404),

$$(14.102) \quad A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T = \text{constant};$$

from the second, we know that  $\mathbf{h}$  has a direction fixed in space and also a constant magnitude, so that

$$(14.103) \quad A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = h^2 = \text{constant}.$$

Let us draw through  $O$  a line  $OP$  in the fixed direction of  $\mathbf{h}$  (Fig. 145); this is called the *invariable line*. Let  $OQ$  represent the angular velocity  $\omega$  at any instant. Drop the perpendicular  $QN$  on  $OP$ ; then  $ON = \omega \cdot \mathbf{h}/h$ . But, by (14.101) and (14.102),

$$(14.104) \quad \omega \cdot \mathbf{h} = 2T',$$

and so

$$(14.105) \quad ON = \frac{2T}{h} = \text{constant}.$$

Thus  $N$  is a fixed point during the motion, and so the plane through  $N$ , perpendicular to the invariable line  $OP$ , is a fixed plane; it is called the *invariable plane*. The extremity  $Q$  of the angular velocity vector  $\omega$  moves on the invariable plane.

Let us now take the point of view of an observer who moves with the body. (This is what we do in our daily lives, for we live on a rotating earth but regard a point on the earth's surface as "fixed.") To such an observer, the vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are fixed, but *both* the vectors  $\mathbf{h}$  and  $\omega$  are changing. If  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are taken as coordinate axes and the extremity of the vector  $\omega$  is given coordinates  $x, y, z$ , then

$$x = \omega_1, y = \omega_2, z = \omega_3.$$

By virtue of (14.102) and (14.103), we have

$$(14.106) \quad Ax^2 + By^2 + Cz^2 = 2T, \quad A^2x^2 + B^2y^2 + C^2z^2 = h^2.$$

In fact, *to an observer moving with the body, the extremity  $Q$  of the angular velocity vector  $\omega$  describes a curve which is the intersection of the two ellipsoids (14.106), fixed in the body.*

The first of these two ellipsoids is similar to the momental ellipsoid and has the same axes; it is called the *Poinsot ellipsoid*.

The invariable plane is fixed in space, but to the observer moving with the

body it is a moving plane. It touches a sphere of radius  $ON$ , but it has another remarkable property: *the invariable plane touches the Poinsoth ellipsoid at the extremity of the angular velocity vector.*

To see this, we note that the tangent plane to the Poinsoth ellipsoid at the point  $(\omega_1, \omega_2, \omega_3)$  is

$$(14.107) \quad A\omega_1x + B\omega_2y + C\omega_3z = 2T.$$

So the direction ratios of the normal to the ellipsoid at this point are

$$A\omega_1, B\omega_2, C\omega_3.$$

But these are precisely the components of angular momentum; hence the normal to the Poinsoth ellipsoid at the extremity of the angular velocity vector is parallel to the angular momentum vector, i.e., parallel to  $OP$ . This proves the result.

As we have indicated, there are two different points of view:

- (i) the point of view of an observer  $S$  fixed in space;
- (ii) the point of view of an observer  $S'$  fixed in the body.

It is confusing to try to look at things simultaneously from the two points of view. We shall clarify the situation by taking them up separately.

The observer  $S$ , fixed in space, cuts away (in his imagination) all the body except an ellipsoid—the Poinsoth ellipsoid. He fixes his attention on this moving ellipsoid and on a fixed plane (the invariable plane). As the body moves, the ellipsoid always touches the plane. It actually *rolls* on the plane, since it has an angular velocity vector which passes through the point of contact of the ellipsoid and the plane. This is a fairly complicated type of motion; it becomes quite simple, however, when the Poinsoth ellipsoid is a surface of revolution, as we shall see later. But it may in any case be visualized by thinking of the invariable plane as a sheet of paper and the Poinsoth ellipsoid as an inked surface. In the course of the motion a curve is thus drawn in ink on the invariable plane. On joining the fixed point  $O$  to the points on this curve, we get the *space cone* (cf. Sec. 11.2).

On the other hand, the observer  $S'$ , fixed in the body, turns his attention to the two ellipsoids (14.106), fixed as far as he is concerned, and in particular to their curve of intersection. The angular velocity vector traces out a cone (the *body cone*), formed by joining the fixed point  $O$  to this curve.

The two points of view are brought into contact by the general result: the body cone rolls on the space cone. The difference between the two is this:  $S$  regards the space cone as fixed, but  $S'$  regards the body cone as fixed.

$$(14.110) \quad \omega_1^2 = P - Q\omega_2^2, \quad \omega_3^2 = R - S\omega_2^2,$$

where  $P, Q, R, S$  are positive expressions involving  $A, B, C, T, h$ . Substitution in the second equation of (14.108) gives

$$(14.111) \quad \dot{\omega}_2^2 = \left( \frac{C - A}{B} \right)^2 (P - Q\omega_2^2)(R - S\omega_2^2).$$

This equation is of the same form as (13.312) and may be treated in the same way. Thus, from (14.111), we get

$$(14.112) \quad \left( \frac{d\xi}{d\tau} \right)^2 = (1 - \xi^2)(1 - k^2\xi^2),$$

where

$$(14.113) \quad \xi = \frac{\omega_2}{\beta}, \quad \tau = pt,$$

the constants  $\beta, p, k$  being positive functions of  $A, B, C, T, h$  with  $k < 1$ . Hence,

$$(14.114) \quad \omega_2 = \beta \operatorname{sn} [p(t - t_0)],$$

where  $t_0$  is a constant of integration. Substitution in (14.110) gives either

$$(14.115a) \quad \omega_1 = \alpha \operatorname{dn} [p(t - t_0)], \quad \omega_3 = \gamma \operatorname{cn} [p(t - t_0)],$$

or

$$(14.115b) \quad \omega_1 = \alpha \operatorname{cn} [p(t - t_0)], \quad \omega_3 = \gamma \operatorname{dn} [p(t - t_0)],$$

where  $\alpha$  and  $\gamma$  are functions of  $A, B, C, T, h$ , determined except for sign. When we substitute in (14.108), we find that  $\alpha\beta\gamma$  is negative. For definiteness, we may make  $\alpha$  positive by suitable choice of the sense of the vector  $\mathbf{i}$ ; then  $\gamma$  is negative.

That is the outline of the method of finding  $\omega_1, \omega_2, \omega_3$ , as functions of  $t$ . The completion of the argument consists in filling in the algebraic details. Care must be taken in selecting the constants  $\beta, p, k$ , so that  $k$  is less than unity; it becomes necessary to distinguish between the two cases (a)  $h^2 > 2BT$ , and (b)  $h^2 < 2BT$ . In the former case, we arrive at (14.115a), in the latter at (14.115b). We leave it to the reader to verify the following results.

CASE (a):  $h^2 > 2BT$ .

$$(14.116a) \quad \begin{cases} \alpha = \sqrt{\frac{h^2 - 2CT}{A(A - C)}}, \\ \beta = \sqrt{\frac{2AT - h^2}{B(A - B)}}, \\ \gamma = -\sqrt{\frac{2AT - h^2}{C(A - C)}}, \end{cases} \quad \begin{aligned} p &= \sqrt{\frac{(h^2 - 2CT)(A - B)}{ABC}}, \\ k &= \sqrt{\frac{B - C}{A - B} \cdot \frac{2AT - h^2}{h^2 - 2CT}}. \end{aligned}$$

CASE (b):  $h^2 < 2BT$ .

$$(14.116b) \quad \begin{cases} \alpha = \sqrt{\frac{h^2 - 2CT}{A(A - C)}}, \\ \beta = \sqrt{\frac{h^2 - 2CT}{B(B - C)}}, \\ \gamma = -\sqrt{\frac{2AT - h^2}{C(A - C)}}, \end{cases} \quad \begin{aligned} p &= \sqrt{\frac{(2AT - h^2)(B - C)}{ABC}}, \\ k &= \sqrt{\frac{A - B}{B - C} \cdot \frac{h^2 - 2CT}{2AT - h^2}}. \end{aligned}$$

In establishing these results, the following identity is useful:

$$(14.117) \quad (B - C)(h^2 - 2AT) + (C - A)(h^2 - 2BT) + (A - B)(h^2 - 2CT) = 0.$$

But the determination of  $\omega_1, \omega_2, \omega_3$  as functions of  $t$  does not complete the solution of the problem. We should be able to tell, from given initial conditions, the position of the body at any time. To do this, we specify the directions of the triad  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , relative to a triad  $\mathbf{I}, \mathbf{J}, \mathbf{K}$ , fixed in space, by means of the Eulerian angles  $\theta, \phi, \psi$ . Then, by (11.202),

$$(14.118) \quad \begin{cases} \omega_1 = \sin \psi \dot{\theta} - \sin \theta \cos \psi \dot{\phi}, \\ \omega_2 = \cos \psi \dot{\theta} + \sin \theta \sin \psi \dot{\phi}, \\ \omega_3 = \cos \theta \dot{\phi} + \dot{\psi}. \end{cases}$$

If we substitute for  $\omega_1, \omega_2, \omega_3$  from (14.114) and (14.115), we obtain three differential equations for  $\theta, \phi, \psi$ . The solution of these equations in this general form presents a formidable problem. It is greatly simplified if we choose the vector  $\mathbf{K}$  in the direction of the invariable line, defined by the constant vector  $\mathbf{h}$ . Then the components of  $\mathbf{h}$  along  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are found by multiplying  $h$  by the direction cosines of  $\mathbf{K}$  relative to  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . These direction cosines are easily found (cf. Fig. 118, page 289) by projecting  $\mathbf{K}$  on  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ; they are

$$-\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta.$$

Hence,



$$(14.119) \quad \begin{cases} A\omega_1 = -h \sin \theta \cos \psi, \\ B\omega_2 = h \sin \theta \sin \psi, \\ C\omega_3 = h \cos \theta. \end{cases}$$

From these equations, we get  $\theta$  and  $\psi$  as functions of  $t$  without any integration, thus:

$$(14.120) \quad \cos \theta = \frac{C\omega_3}{h}, \quad \tan \psi = -\frac{B\omega_2}{A\omega_1}.$$

To find  $\phi$ , we deduce, from the first two of (14.118),

$$(14.121) \quad \sin \theta \dot{\phi} = \omega_2 \sin \psi - \omega_1 \cos \psi,$$

and so  $\phi$  is obtained by a quadrature, since  $\theta, \psi, \omega_1, \omega_2$  are already known as functions of  $t$ .

From the periodic property of the elliptic functions, we see that  $\theta, \sin \psi, \cos \psi, \phi$  are periodic functions of  $t$ ; in general,  $\phi$  does not increase by a multiple of  $2\pi$  in a period, and the motion as a whole is not periodic.

### The case of a body with an axis of symmetry.

When the momental ellipsoid at the fixed point  $O$  has an axis of symmetry, two of the three moments of inertia  $A, B, C$  become equal to one another. This will be the case if the body is a solid of revolution of uniform density, but all that is actually required is the symmetry of the momental ellipsoid. The motion of the body under no forces is then greatly simplified. In fact, the simplification is so great that it is easier to discuss the problem afresh, rather than to apply the formulas of the general case. The motion can be determined, both qualitatively and quantitatively, by the method of Poinsot.

Let us denote the principal moments of inertia at  $O$  by  $A$  and  $C$ ,  $C$  being the *axial moment of inertia* and  $A$  the *transverse moment of inertia*. (This means that  $C$  is the moment of inertia about the axis of symmetry and  $A$  the moment of inertia about any perpendicular line through  $O$ .) The cases  $A > C$  and  $A < C$  differ in some respects, but for the present we may treat them together.

The Poinsot ellipsoid is of revolution. Since its center is fixed and it rolls on the invariable plane, the following facts are obvious:

- (i) The body cone and the space cone are both right circular cones.
- (ii) The angular velocity vector is of constant magnitude ( $\omega = OQ$ ) and makes a constant angle with the invariable line  $OP$  (cf. Fig. 145).
- (iii) The invariable line, the angular velocity vector, and the axis of symmetry are coplanar at every instant.

$\omega$  of constant magnitude, inclined to  $OP$  at a constant angle; hence the plane containing the axis of symmetry and the angular velocity vector rotates about  $OP$  with a constant angular velocity. We shall denote this angular velocity by  $\Omega$ .

There is one more constant of importance. It is the angular velocity of the instantaneous axis about the axis of symmetry, as judged by an observer moving with the body. We shall denote this angular velocity by  $n$ .

We have, in all, the following constants:

$$\alpha, \beta, \omega, \Omega, n.$$

Of these,  $\alpha$  and  $\omega$  are determined by initial conditions;  $\beta$  is given by (14.122). We shall now set up equations to find  $\Omega$  and  $n$ , and at the same time get a clear picture of the motion by considering the space and body cones (Figs. 147a and b).  $OP$  is the invariable line,  $OQ$  the instantaneous axis,  $OR$  the axis of symmetry, and  $QN$ ,  $QM$  are drawn perpendicular to  $OP$ ,  $OR$ , respectively. In each case the space cone is fixed, and the body cone rolls on it. This motion is easy to follow in Fig. 147a. The motion in Fig. 147b may be understood by thinking of what the motion looks like from above, or by making a simple model out of thick paper and working the cones through the fingers in order to reproduce the condition of rolling.

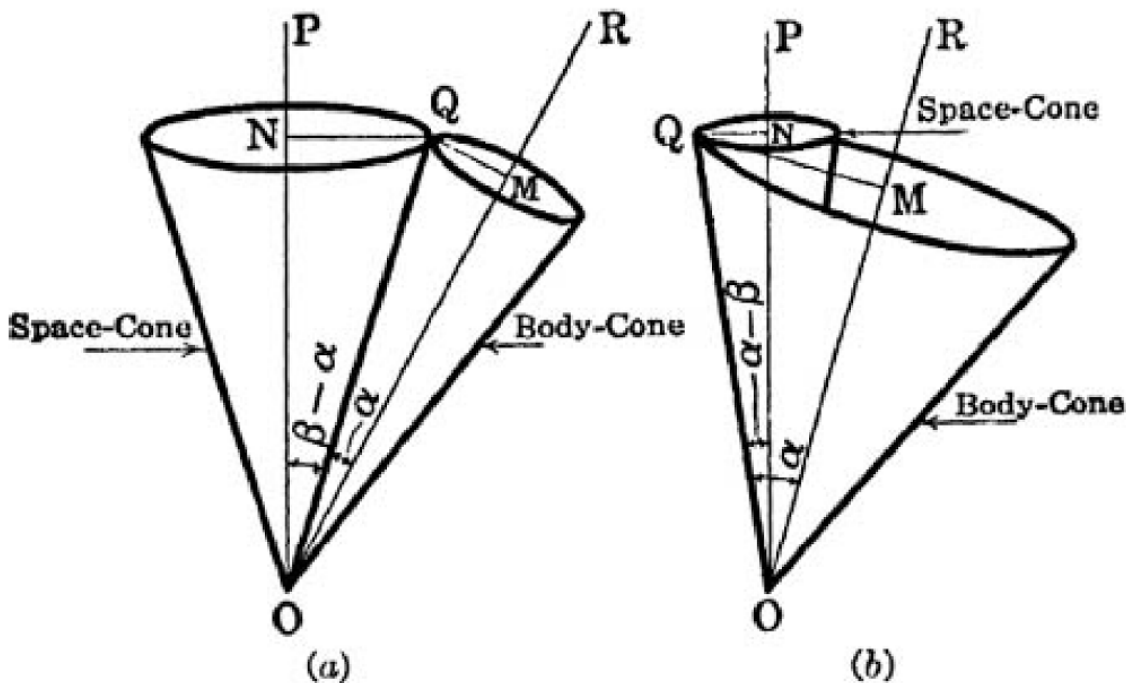


FIG. 147.—(a) The case where  $A > C$ . (b) The case where  $A < C$ .

CASE (a) :  $A > C$ . The line  $OR$  turns about  $OP$  with angular velocity  $\Omega$ . Thus, in time  $dt$ , the point  $M$  receives a displacement

The sense of this rotation  $n$  is direct, when compared with  $\omega$ .

In the case of the earth, which is slightly flattened from the spherical form, we have  $A < C$ , and the ratio  $(C - A)/A$  is small. Thus, case (b) applies, but in an extreme form, since the instantaneous axis is close to the axis of symmetry and  $\alpha$  is small. The angular velocity  $n$  represents the rate at which the celestial pole, or axis of rotation of the earth, moves round the earth's axis of symmetry. For the period, we have approximately

$$\frac{2\pi}{n} = \frac{A}{C - A} \cdot \frac{2\pi}{\omega}.$$

If the sidereal day is taken as unit of time, then  $2\pi/\omega = 1$ , and calculation gives the value 305 for the period. This prediction is in poor agreement with observation; for though a rotation of this sort is observed, its period is about 440 days.\* The model used (a rigid body) proves at fault here, on account of the elasticity of the earth.

*Exercise.* A circular disk is mounted so that it can turn freely about its center. Its angular velocity vector makes an angle of  $45^\circ$  with its plane and has a magnitude of 20 revolutions per second. Make a rough sketch of the space and body cones, and find  $\Omega$  and  $n$ .

## 14.2. THE SPINNING TOP

The spinning top is the most familiar example of a gyroscopic system. The word “gyroscope” was invented to denote an instrument in which the earth's rotation produced an effect which could be observed. But the word “gyroscope” (or “gyrostat”) is now used for any system in which a rapidly rotating body is so mounted that it may change the direction of its angular velocity vector.

Why does a spinning top not fall down? How does its rapid rotation render it apparently immune to the force of gravity, which makes non-spinning bodies fall? It is difficult to give a simple answer to this question. The only way to explain the phenomenon is to construct the mathematical theory of the top. For our purposes, we shall understand a “top” to mean a rigid body with an axis of symmetry, acted on by the force of gravity. A point on the axis of symmetry is fixed. Thus we idealize the ordinary top by supposing it to terminate in a sharp point (or vertex) and to spin on a floor rough enough to prevent slipping.

### **Steady precession of a top.**

The motion of any rigid body with a fixed point  $O$  satisfies the equation

$$a\mathbf{k} \times (-mg\mathbf{K}) = -mga \sin \theta \mathbf{j}$$

about  $O$ , If this is equal to  $\mathbf{G}$ , as given by (14.207), no force other than the weight of the top is required to maintain the motion. Thus *the steady precession takes place under gravity alone if*

$$(14.208) \quad p(Cs - Ap \cos \theta) = mga.$$

This is a single equation connecting the three constants  $\theta$ ,  $p$ ,  $s$ . There is, therefore, a doubly infinite set of steady precessions corresponding to arbitrary values of two out of the three constants. It is not, however, possible to assign completely arbitrary values of two of the constants; these values must be such that (14.208) yields a real value for the third constant.

If we see a top spinning,  $\theta$  and  $p$  are easy to observe. In terms of them,  $s$  is given by

$$(14.209) \quad s = \frac{mga}{Cp} + \frac{Ap \cos \theta}{C}.$$

We note that, if the precession is small, the spin is great and is given approximately by

$$(14.210) \quad s = \frac{mga}{Cp}.$$

This is a very simple and useful formula.

*Exercise.* A disk, 6 inches in diameter, is mounted on the end of a light rod 1 inch long and spins rapidly. It precesses once in 15 seconds. Find approximately the spin, in revolutions per second, and the velocity of a point on the edge of the disk.

### **General motion of a top.**

To discuss the general motion of a top, we shall use the same notation for the constants of the top as that used above.

$$(14.224) \quad A(\dot{\theta}^2 + \phi^2 \sin^2 \theta) + Cs^2 = 2(E - mga \cos \theta).$$

We have in (14.221) and (14.224) two equations to determine  $\theta$  and  $\phi$  as functions of the time.

It is convenient to write  $Cs = \beta$ . Then our two equations read

$$(14.225) \quad \begin{cases} A\phi \sin^2 \theta = \alpha - \beta \cos \theta \\ A(\dot{\theta}^2 + \phi^2 \sin^2 \theta) + \frac{\beta^2}{C} = 2(E - mga \cos \theta). \end{cases}$$

The plan is now obvious. We are to substitute for  $\phi$  in the second equation from the first; this will give a differential equation for  $\theta$ . When this is solved, the first equation will give  $\phi$  by a quadrature.

Let us put  $x = \cos \theta$ . On multiplying the second equation in (14.225) by  $\sin^2 \theta$  and substituting for  $\phi$  we obtain for  $x$  the differential equation

$$(14.226) \quad A \left[ \dot{x}^2 + \frac{(\alpha - \beta x)^2}{A^2} \right] + \frac{\beta^2}{C} (1 - x^2) = 2(E - mgax)(1 - x^2).$$

This equation may be written

$$(14.227) \quad \dot{x}^2 = f(x),$$

where

$$(14.228) \quad f(x) = \frac{1}{A} \left( 2E - \frac{\beta^2}{C} - 2mgax \right) (1 - x^2) - \frac{(\alpha - \beta x)^2}{A^2}.$$

This is a cubic in  $x$ , and, by the same form of argument as that used in Sec. 13.3 for the spherical pendulum, we see that it has a graph of the general form shown in [Fig. 150](#); the function  $f(x)$  has three real zeros  $x_1, x_2, x_3$ , such that

$$-1 < x_1 < x_2 < 1 < x_3.$$

(In special cases, we may have one or more signs of equality instead of inequality.) Thus  $f(x)$  may be written

$$(14.229) \quad f(x) = \frac{2mga}{A} (x - x_1)(x - x_2)(x - x_3).$$

Again by the same argument as in Sec. 13.3, the solution of (14.227) is

$$(14.230) \quad \cos \theta = x = x_1 + (x_2 - x_1) \operatorname{sn}^2 [p(t - t_0)],$$

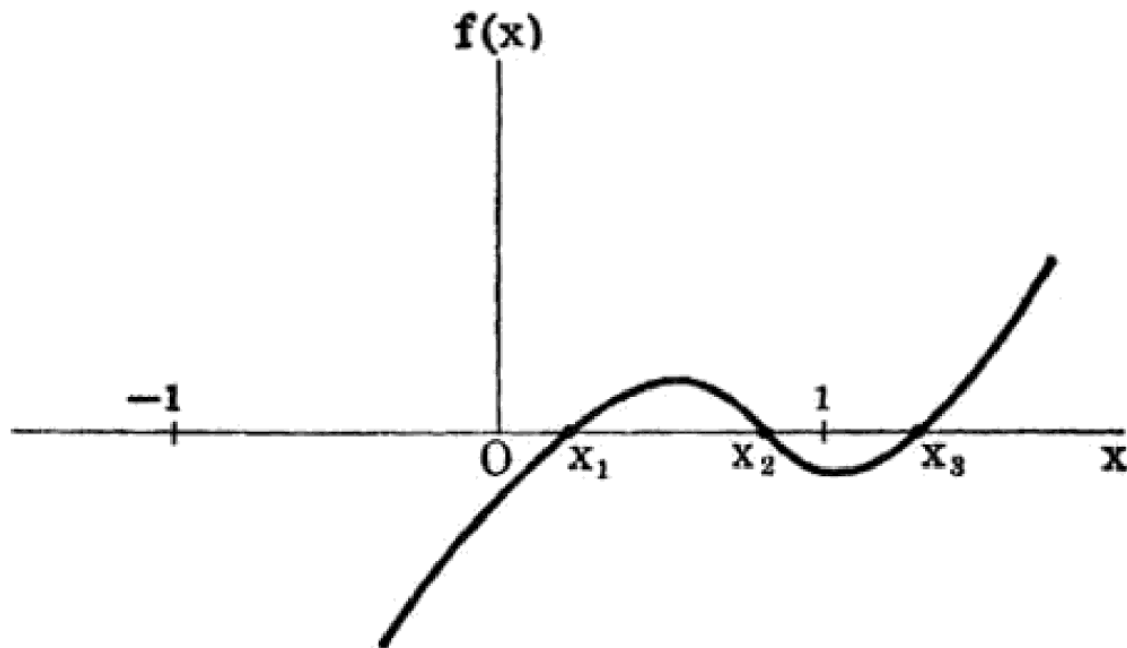


FIG. 150.—Graph of  $f(x)$  for a top in general motion.

where  $p$  and the modulus  $k$  of the elliptic function are given by

$$(14.231) \quad p^2 = \frac{mga(x_3 - x_1)}{2A}, \quad k^2 = \frac{x_2 - x_1}{x_3 - x_1}.$$

The constants  $x_1, x_2, x_3$  are functions of the constants occurring in (14.228), i.e., the constants of the top and  $\alpha, \beta E$ ; the latter are known when the initial position and angular velocity of the top are given.

The complete solution for the motion of the axis of the top is given by (14.230) and

$$(14.232) \quad \phi = \frac{\alpha - \beta x}{A(1 - x^2)}.$$

Since  $x$  is known as a function of  $t$ , this last equation gives  $\phi$  by a quadrature.

This analytic solution does not immediately give a clear idea of the way in which the top behaves. However, we can construct the essential features of the

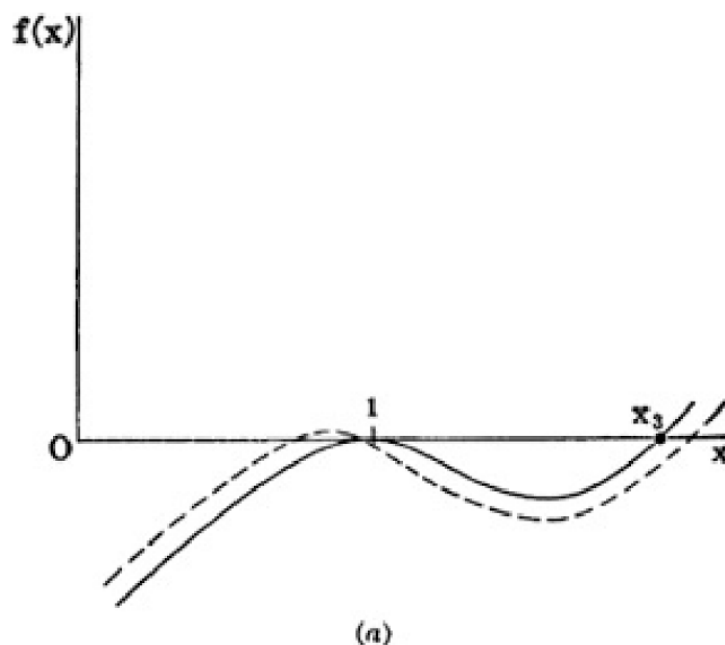


hence, (14.228) gives

$$(14.238) \quad f(x) = (1 - x)^2 \left[ \frac{2mga}{A} (1 + x) - \frac{C^2 s^2}{A^2} \right].$$

We observe that  $x = 1$  is a double zero of  $f(x)$ . Two cases arise: either the third zero of  $f(x)$  is greater than unity, or it is less than unity. The forms of the graph of  $f(x)$  for these two cases are shown in Figs. 152a and b; in terms of the notation used for the zeros of  $f(x)$ , Fig. 152a shows the case where the third zero is  $x_3$ , and Fig. 152b the case where it is  $x_1$ .

When the top suffers a small disturbance, the graph of  $f(x)$  for the disturbed motion will not be the same as that for the sleeping top. The difference will be small, however, since only small changes in the constants  $\alpha$ ,  $\beta$ ,  $E$  can result from a small disturbance. The broken curves in Figs. 152a and b indicate the way in which the graphs of  $f(x)$  are modified by a small disturbance. In general, the three zeros of  $f(x)$  will become distinct—two of them must, of course, lie in the range  $(-1, 1)$ , and the third must exceed unity.





## Stability of a spinning projectile.

It is a well-known fact that an elongated projectile acquires *stability* from the spin imparted to it by the rifling in the gun. By this we mean that it does not turn broadside on to its direction of motion when in flight, nor does it tumble as a nonspinning projectile often does. We shall now use our theory of the motion of a top in an attempt to explain this spin stabilization.

In Sec. 6.2 we discussed the motion of a projectile in a resisting medium, the projectile being treated as a particle. The problem becomes much more complicated when the projectile is treated as a rigid solid of revolution, subject to gravity and to the aerodynamic force system due to the pressure of the air. We cannot enter here into this general problem; instead, following the older writers on ballistics, we shall make some drastic simplifications.

Thus, we shall assume that the aerodynamic force system is equipollent to a single force (the drag) with fixed direction and constant magnitude ( $R$ ), intersecting the axis of the projectile at a fixed point (the center of pressure). We can now use Fig. 149, with suitable changes, for the discussion of the motion of the projectile relative to its mass center. Let  $O$  be the mass center,  $\mathbf{K}$  a fixed unit vector opposed to the direction of the drag, and  $\mathbf{k}$  a unit vector along the axis of the projectile. The point  $D$  is taken to be the center of pressure; and the force at  $D$ , i.e.,  $-mg\mathbf{K}$  in the case of the top, is now to be replaced by  $-R\mathbf{K}$ . The weight of the projectile acts through  $O$ , and so the total moment about  $O$  is due to the aerodynamic forces alone; it is

$$(14.240) \quad \mathbf{G} = -Ra \sin \theta \mathbf{j},$$

where  $a$  is the distance of the center of pressure in front of the mass center and  $\theta$  the angle between the axis of the projectile and the direction of the drag reversed.

Since the fundamental equation (12.209) for motion relative to the mass center is of the same form as (14.218) and the expression (14.240) for  $\mathbf{G}$  is of the same form as (14.217) with the constant  $mga$  changed to  $Ra$ , we can apply to the motion of the projectile the same mathematical analysis as we applied to the motion of the top. In particular, if the projectile is moving along its axis, the fixed direction of the drag will also lie on this line, and we have what is essentially a sleeping top. Then (14.239) yields the condition for stability of the spinning projectile, i.e., the condition that the projectile, if slightly disturbed, will not develop large oscillations. This condition is

$$(14.241) \quad s^2 > \frac{4ARa}{C^2},$$

where  $s$  is the spin of the projectile,  $A$  the transverse moment of inertia at  $O$  (i.e., at the mass center), and  $C$  the axial moment of inertia.

### 14.3. GYROSCOPES

#### The stability of a gyroscope.

Let us suppose that a gyroscope (i.e., a rigid body with an axis of symmetry) is mounted in a Cardan's suspension (Fig. 144), so that its mass center is fixed. It is set spinning about its axis of symmetry with a great angular velocity  $s$ . Now let an impulsive couple  $\hat{\mathbf{G}}$  be applied to the gyroscope. The instantaneous change in angular momentum is [cf. (12.502)]

$$(14.301) \quad \Delta \mathbf{h} = \hat{\mathbf{G}}.$$

We have then a vector diagram as in Fig. 153. The vector  $\vec{OA}$  is  $\mathbf{h}$ , the angular momentum before the impulsive couple was

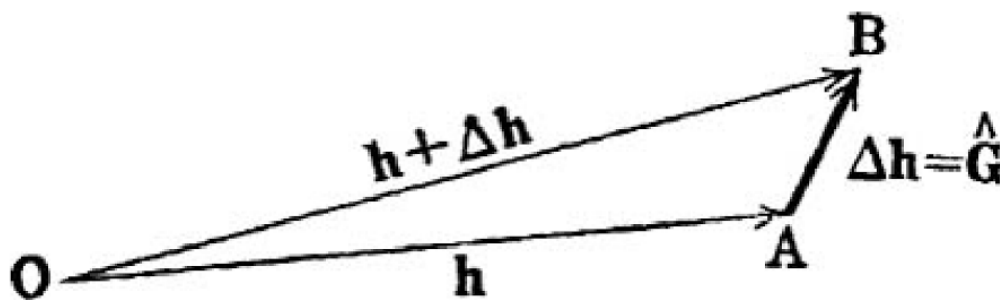


FIG. 153.—Change in angular momentum due to an impulsive couple.

applied. It is made long, because  $s$  (and consequently  $h$ ) is assumed to be large. The vector  $\vec{AB}$  is  $\Delta \mathbf{h}$ , and  $\vec{OB}$  represents the final angular momentum. It is clear that the angle  $AOB$  is small; in fact, it tends to zero as  $s$  tends to infinity. Thus, the application of an impulsive couple to a rapidly spinning gyroscope makes only a small change in the direction of the angular momentum vector. It is easily seen that the corresponding change in the direction of the angular velocity vector is also small.

This simple result illustrates the stability which a rapid rotation imparts to a body. The gyroscope shows, as it were, an unwillingness to alter the direction of its axis. When it does yield, it does so in a manner which continues to cause surprise even to those familiar with the theory.

#### The gyroscopic couple.

In Fig. 154a,  $O$  is a fixed point on the axis of a gyroscope, and  $\mathbf{j}$  a unit

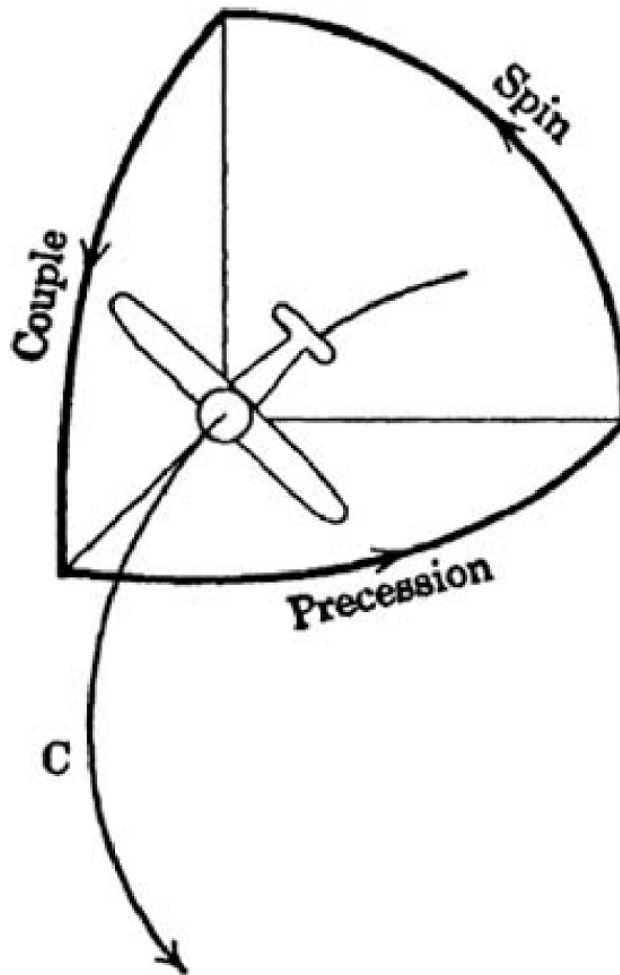


FIG. 155.—Airplane turning.

If the pilot flies the airplane with a rotary engine in the same way as he would fly a similar airplane with a stationary engine, the couple required to maintain the steady flight in a horizontal circle will not be present. Since the couple is one which tends to depress the nose, in its absence the nose will rise. What will happen after the initial lift takes place is a complicated question, not covered by the present simple theory.

## The gyrocompass.

If the spinning of the earth on its axis were much faster than it actually is, it would be a simple matter to devise a mechanism by which the true north could be found on a ship at sea. However, the earth's rotation is so slow that an apparatus of great delicacy is required, in order that a minute effect may not be wiped out by frictional resistances. The modern gyrocompass is such a piece of apparatus. The simple system which we shall discuss is much less elaborate than the gyrocompass as it is actually constructed.\* However, the basic fact that a spinning gyroscope enables us to find the north is demonstrated by a discussion of an ideally simple gyrocompass.

In Fig. 156,  $PQ$  is part of the earth's axis, drawn from south to north. The point  $O$  is on the earth's surface at latitude  $\lambda$ . Thus the horizontal line, drawn

due north from  $O$ , is inclined to the earth's axis at an angle  $\lambda$ ; this line is  $OQ$  in the diagram. The unit vector  $\mathbf{K}$  is parallel to  $PQ$ .

A gyroscope is mounted in a Cardan's suspension (Fig. 144) so that its mass center lies at  $O$ . But it is not left free to turn about  $O$ ; one pair of the bearings in the suspension is locked, so that the axis of the gyroscope can move only in the horizontal plane at  $O$ . The unit vector  $\mathbf{k}$  lies along the axis of the gyroscope, making with  $OQ$  a variable angle  $\theta$ ;  $\mathbf{i}$  is perpendicular to  $\mathbf{k}$  in the horizontal plane, and  $\mathbf{j}$  is vertical (i.e., coplanar with  $PQ$ ,  $OQ$  and perpendicular to  $OQ$ ). The gyroscope itself is not shown in the diagram; the curve is a unit circle in the horizontal plane.

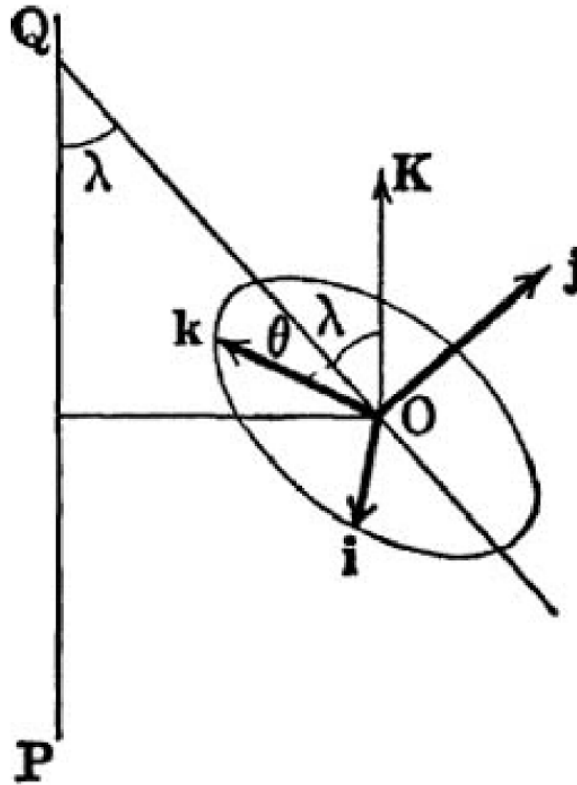


FIG. 156.—Vector diagram for gyrocompass.

The angular velocity of the triad  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  is made up of the angular velocity of the earth, which we may write  $\Omega \mathbf{K}$ , and an angular velocity due to change in  $\theta$ , in fact,  $\dot{\theta} \mathbf{j}$ . Since

$$\mathbf{K} = -\sin \theta \cos \lambda \mathbf{i} + \sin \lambda \mathbf{j} + \cos \theta \cos \lambda \mathbf{k},$$

the angular velocity of the triad is

$$(14.307) \quad \boldsymbol{\omega}' = -\Omega \sin \theta \cos \lambda \mathbf{i} + (\dot{\theta} + \Omega \sin \lambda) \mathbf{j} + \Omega \cos \theta \cos \lambda \mathbf{k}.$$

The angular velocity  $\boldsymbol{\omega}$  of the gyroscope differs from this only in the third

component; thus,

$$(14.308) \quad \boldsymbol{\omega} = -\Omega \sin \theta \cos \lambda \mathbf{i} + (\dot{\theta} + \Omega \sin \lambda) \mathbf{j} + s \mathbf{k},$$

where  $s$  is the axial spin, including a component of the earth's rotation. The angular momentum is

$$(14.309) \quad \mathbf{h} = -A \Omega \sin \theta \cos \lambda \mathbf{i} + A(\dot{\theta} + \Omega \sin \lambda) \mathbf{j} + C s \mathbf{k},$$

where  $A$  and  $C$  are, respectively, the transverse and axial moments of inertia.

To keep the axis of the gyroscope in the horizontal plane, the bearings of the suspension must exert a couple  $\mathbf{G}$  on it. Since the bearings are assumed to be smooth, no work is done by this couple in rotations about either  $\mathbf{j}$  or  $\mathbf{k}$ . Hence,  $\mathbf{G}$  is perpendicular to these vectors, and we may write

$$(14.310) \quad \mathbf{G} = G \mathbf{i},$$

where  $G$  may be either positive or negative.

The fundamental equation  $\dot{\mathbf{h}} = \mathbf{G}$  gives

$$(14.311) \quad -A \Omega \cos \theta \cos \lambda \dot{\theta} \mathbf{i} + A \ddot{\theta} \mathbf{j} + C \dot{s} \mathbf{k} + \boldsymbol{\omega}' \times \mathbf{h} = G \mathbf{i}.$$

We now pick out the  $\mathbf{j}$  and  $\mathbf{k}$  components of this equation and so obtain two scalar equations for  $s$  and  $\theta$ :

$$(14.312) \quad \begin{cases} A \ddot{\theta} + C s \Omega \cos \lambda \sin \theta - A \Omega^2 \sin \theta \cos \theta \cos^2 \lambda = 0, \\ C \dot{s} = 0. \end{cases}$$

The second equation shows that the spin  $s$  is constant. With  $\Omega^2$  neglected, the first equation may be written

$$(14.313) \quad \ddot{\theta} + n^2 \sin \theta = 0,$$

where

$$(14.314) \quad n = \sqrt{\frac{C s \Omega \cos \lambda}{A}}.$$

Now (14.313) is the equation of motion of a simple pendulum. It is possible for the axis of the gyroscope to go right round the horizontal circle, but if the initial values of  $\theta$  and  $\dot{\theta}$  are small, the motion will be oscillatory. The important



fact is this: *The axis of the gyroscope oscillates symmetrically about the direction  $\theta = 0$ .* Hence, by bisecting the angle of swing, we may find the north. The gyroscope therefore acts as a *gyrocompass*, indicating the *true* north; the magnetic compass, of course, indicates the *magnetic* north.

For small oscillations, the periodic time of swing for the gyrocompass is

$$(14.315) \quad \tau = \frac{2\pi}{n} = 2\pi \sqrt{\frac{A}{Cs\Omega \cos \lambda}}.$$

Since  $\Omega$  is so small (1 revolution per sidereal day =  $2\pi/86,164$  radians per second), the spin of the gyroscope ( $s$ ) must be given a large value in order to make  $\tau$  reasonably small. If  $\lambda = \pm \frac{1}{2}\pi$ , the periodic time becomes infinite, and the gyrocompass fails to function; but that is only to be expected, for the points in question are the North and South Poles.

*Exercise* Assuming the mass of the gyroscope concentrated in a thin ring, find the number of revolutions per second required for a periodic time of 10 seconds at latitude  $45^\circ$ .

#### 14.4. GENERAL MOTION OF A RIGID BODY

The general motion of a rigid body consists of (i) motion of the mass center and (ii) motion relative to the mass center. The equations governing these have been given in Sec. 12.4. But it would be wrong to suppose that the determination of the general motion always splits into two parts—a problem in particle dynamics and a problem in the dynamics of a body with a fixed point. Constraints make the two problems interlock, and complications arise. We cannot give a general plan for the solution of all such problems but shall determine the motions of two systems as examples.

##### The motion of a billiard ball.

A billiard ball is struck by a cue. At time  $t = 0$ , we suppose that the ball is in contact with the table; its center has a horizontal velocity  $q_0$ , and the ball has an angular velocity  $\omega_0$ . We wish to find the subsequent motion of the ball.

If the table were perfectly smooth, the center of the ball would retain the velocity  $q_0$ , and the angular velocity  $\omega_0$  would also be retained. But we shall assume the table to be rough, with a coefficient of kinetic friction  $\mu$ .

At a general time  $t$ , the center  $O$  has a horizontal velocity  $q$ , and the ball has an angular velocity  $\omega$ . Let  $\mathbf{K}$  be a unit vector drawn vertically upward (Fig. 157). The reaction of the table on the ball at the point of contact  $P$  may be

written

$$\mathbf{F} + R\mathbf{K},$$

where  $R$  is the magnitude of the normal reaction and  $\mathbf{F}$  the force of friction;  $\mathbf{F}$ , of course, acts horizontally. The weight is  $-mg\mathbf{K}$ , where  $m$  is the mass of the ball. Thus the equation of motion of the center is, by (12.410),

$$(14.401) \quad m\dot{\mathbf{q}} = \mathbf{F} + (R - mg)\mathbf{K}.$$

But, since the ball remains in contact with the table, the acceleration of the center has no vertical component. Thus  $R = mg$ , and we have

$$(14.402) \quad m\dot{\mathbf{q}} = \mathbf{F}.$$

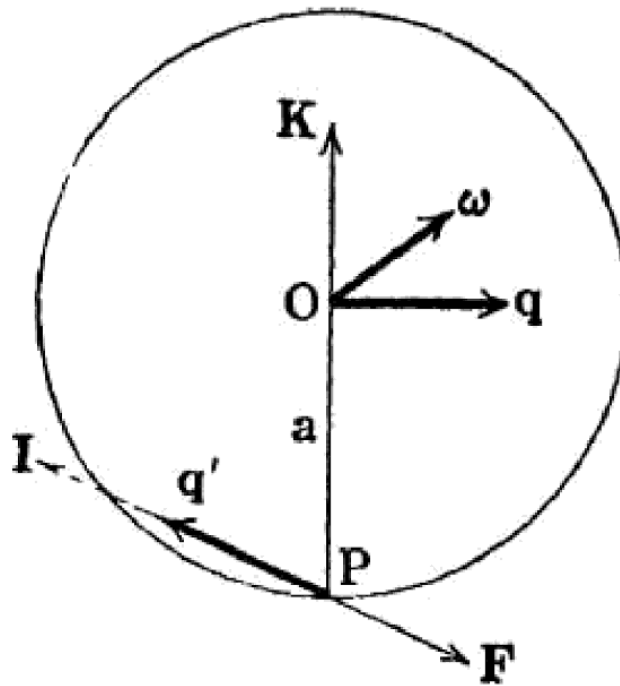


FIG. 157.—Ball slipping on a table.

Since every axis at  $O$  is a principal axis of inertia, the angular momentum about  $O$  is  $\mathbf{h} = mk^2\boldsymbol{\omega}$ , where  $k$  is the radius of gyration about a diameter. Thus the equation for motion relative to  $O$  is, by (12.411),

$$(14.403) \quad mk^2\dot{\boldsymbol{\omega}} = -a\mathbf{K} \times (\mathbf{F} + R\mathbf{K}) = -a\mathbf{K} \times \mathbf{F},$$

where  $a$  is the radius of the ball.

In the vector equations (14.402) and (14.403), there are actually five scalar equations. There are seven unknowns, viz., two components of  $\mathbf{q}$ , three components of  $\boldsymbol{\omega}$ , and two components of  $\mathbf{F}$ . Thus, two more equations are



required; they are furnished by the law of kinetic friction, as long as there is slipping between the ball and the table. This law tells us that  $\mathbf{F}$  acts in a direction opposite to the velocity of the particle of the ball at  $P$ , and that

$$(14.404) \quad F = \mu R = \mu mg.$$

Thus,

$$(14.405) \quad \mathbf{F} = -\mu mg \frac{\mathbf{q}'}{q'},$$

where  $\mathbf{q}'$  is the velocity of the particle at  $P$ , given by

$$(14.406) \quad \mathbf{q}' = \mathbf{q} + \boldsymbol{\omega} \times (-a\mathbf{K}).$$

Hence, using (14.402) and (14.403), we get

$$(14.407) \quad \begin{aligned} m\dot{\mathbf{q}}' &= \mathbf{F} + \frac{a^2}{k^2} (\mathbf{K} \times \mathbf{F}) \times \mathbf{K} \\ &= \left(1 + \frac{a^2}{k^2}\right) \mathbf{F}, \end{aligned}$$

since  $\mathbf{K} \cdot \mathbf{F} = 0$ . Thus, by (14.405) and (14.407), the derivative of  $\mathbf{q}'$  has a direction opposed to  $\mathbf{q}'$ . This implies that, as long as slipping is taking place, the vector  $\mathbf{q}'$  has a fixed direction.

Let  $\mathbf{I}$  be a unit vector in this fixed direction. Then

$$(14.408) \quad \mathbf{q}' = q'\mathbf{I}, \quad \mathbf{F} = -\mu mg\mathbf{I},$$

and, by (14.407), the magnitude of  $q'$  changes according to the equation

$$(14.409) \quad \dot{q}' = -\mu g \left(1 + \frac{a^2}{k^2}\right).$$

Hence,

$$(14.410) \quad q' = q'_0 - \mu g t \left(1 + \frac{a^2}{k^2}\right),$$

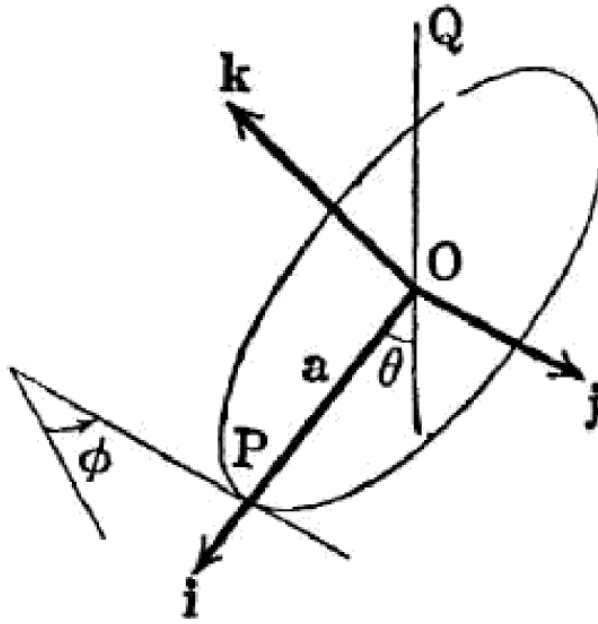


FIG. 158.—Disk rolling on a plane.

For the velocity of the center and the angular velocity of the disk, we may write

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad \boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}.$$

These two vectors are not independent, because the particle at  $P$  is instantaneously at rest. This gives the condition

$$\mathbf{q} + \boldsymbol{\omega} \times a\mathbf{i} = 0,$$

where  $a$  is the radius of the disk; or, in scalar form,

$$(14.414) \quad u = 0, \quad v + a\omega_3 = 0, \quad w - a\omega_2 = 0.$$

These equations determine  $\mathbf{q}$  when  $\boldsymbol{\omega}$  is known.

Now the angular velocity  $\boldsymbol{\Omega}$  of the triad arises solely from changes in  $\theta$  and  $\phi$ . The former gives an angular velocity  $-\dot{\theta}\mathbf{j}$ , and the latter an angular velocity  $\dot{\phi}$  about  $OQ$ , the vertical through  $O$ . Thus,

$$(14.415) \quad \boldsymbol{\Omega} = -\cos \theta \dot{\phi}\mathbf{i} - \dot{\theta}\mathbf{j} + \sin \theta \dot{\phi}\mathbf{k}.$$

But the angular velocities of the disk and the triad differ only in the  $\mathbf{k}$  component. Therefore

$$(14.416) \quad \omega_1 = -\cos \theta \dot{\phi}, \quad \omega_2 = -\dot{\theta}.$$

For the reaction of the ground, we write

$$(14.417) \quad \mathbf{R} = R_1 \mathbf{i} + R_2 \mathbf{j} + R_3 \mathbf{k}.$$

By (12.410) and (12.411), the two vector equations of motion are

$$(14.418) \quad \begin{cases} m\dot{\mathbf{q}} = \mathbf{R} + mg(\cos \theta \mathbf{i} - \sin \theta \mathbf{k}), \\ \dot{\mathbf{h}} = a\mathbf{i} \times \mathbf{R}, \end{cases}$$

where  $m$  is the mass of the disk and  $\mathbf{h}$  is the angular momentum about  $O$ ; we have

$$\mathbf{h} = A\omega_1 \mathbf{i} + A\omega_2 \mathbf{j} + C\omega_3 \mathbf{k},$$

$A$  and  $C$  being transverse and axial moments of inertia at  $O$ .

Since, by (12.306),

$$(14.419) \quad \dot{\mathbf{q}} = \dot{u}\mathbf{i} + \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + \boldsymbol{\Omega} \times \mathbf{q},$$

the first of (14.418) gives the three scalar equations

$$(14.420) \quad \begin{cases} m(\dot{u} - \theta w - \sin \theta \phi v) = R_1 + mg \cos \theta, \\ m(\dot{v} + \sin \theta \phi u + \cos \theta \phi w) = R_2, \\ m(\dot{w} - \cos \theta \phi v + \theta u) = R_3 - mg \sin \theta. \end{cases}$$

By (14.414) and (14.416), we eliminate  $u, v, w$  and obtain

$$(14.421) \quad \begin{cases} ma(\dot{\theta}^2 + \sin \theta \phi \omega_3) = R_1 + mg \cos \theta, \\ -ma(\dot{\omega}_3 + \cos \theta \theta \phi) = R_2, \\ -ma(\ddot{\theta} - \cos \theta \phi \omega_3) = R_3 - mg \sin \theta. \end{cases}$$

Turning now to the second of (14.418), we have

$$(14.422) \quad \dot{\mathbf{h}} = A\dot{\omega}_1 \mathbf{i} + A\dot{\omega}_2 \mathbf{j} + C\dot{\omega}_3 \mathbf{k} + \boldsymbol{\Omega} \times \mathbf{h},$$

and so we get the three scalar equations

$$(14.423) \quad \begin{cases} A\dot{\omega}_1 - C\dot{\theta}\omega_3 - A \sin \theta \phi \omega_2 = 0, \\ A\dot{\omega}_2 + A \sin \theta \phi \omega_1 + C \cos \theta \phi \omega_3 = -aR_3, \\ C\dot{\omega}_3 - A \cos \theta \phi \omega_2 + A\dot{\theta}\omega_1 = aR_2. \end{cases}$$

By (14.416), these become

$$(14.424) \quad \begin{cases} A \frac{d}{dt} (\cos \theta \dot{\phi}) + C \dot{\theta} \omega_3 - A \sin \theta \dot{\theta} \dot{\phi} = 0, \\ A \ddot{\theta} + A \sin \theta \cos \theta \dot{\phi}^2 - C \cos \theta \dot{\phi} \omega_3 = a R_3, \\ C \dot{\omega}_3 = a R_2. \end{cases}$$

Associating these equations with (14.421), we have six equations for the six unknowns  $\theta$ ,  $\phi$ ,  $\omega_3$ ,  $R_1$ ,  $R_2$ ,  $R_3$ . They are the equations (12.412) applied to our special problem.

Before proceeding to discuss the stability of the disk rolling straight ahead, let us consider simple steady motions satisfying the equations (14.421) and (14.424).

The most obvious solution is

$$(14.425) \quad \begin{cases} \theta = 0, & \phi = \text{constant}, & \omega_3 = \text{constant}, \\ R_1 = -mg, & R_2 = 0, & R_3 = 0. \end{cases}$$

This is the straight-ahead motion, in which the plane of the disk is vertical. Another simple motion is given by

$$(14.426) \quad \theta = \text{constant}, \quad \dot{\phi} = \text{constant}, \quad \omega_3 = \text{constant}.$$

The corresponding reactions are, by (14.421),

$$(14.427) \quad \begin{cases} R_1 = m(a \sin \theta \dot{\phi} \omega_3 - g \cos \theta), \\ R_2 = 0, \\ R_3 = m(a \cos \theta \dot{\phi} \omega_3 + g \sin \theta). \end{cases}$$

When we substitute in (14.424), the satisfaction of these equations requires

$$(14.428) \quad (C + ma^2) \cos \theta \dot{\phi} \omega_3 + mga \sin \theta = A \sin \theta \cos \theta \dot{\phi}^2;$$

this condition must be satisfied by the constant values of  $\theta$ ,  $\phi$ ,  $\omega_3$ , in order that the steady motion may exist. It is, of course, a rolling in a circular path.

*Exercise.* If, in the motion given by (14.426),  $\theta$  and  $\phi$  are small, show that the time taken to complete the circular path is approximately

$$2\pi\omega_3(C + ma^2)/(mga\theta).$$

Let us now discuss the stability of the rolling disk. We suppose the disk to be slightly disturbed from the steady motion given by (14.425). In the

disturbed state the following quantities are assumed to be small, since they vanish in the steady motion:

$$(14.429) \quad \theta, \dot{\theta}, \ddot{\theta}, \phi, \dot{\phi}, \dot{\omega}_3, R_1 + mg, R_2, R_3.$$

With only first-order terms retained, (14.421) and (14.424) become

$$(14.430) \quad \begin{cases} 0 = R_1 + mg, & A\dot{\phi} + C\dot{\theta}\omega_3 = 0, \\ ma\dot{\omega}_3 = -R_2, & A\ddot{\theta} - C\dot{\phi}\omega_3 = aR_3, \\ ma\ddot{\theta} - ma\dot{\phi}\omega_3 = -R_3 + mg\theta, & C'\dot{\omega}_3 = aR_2. \end{cases}$$

From the second and last equations we see that  $\omega_3 = \text{constant}$ . Elimination of  $\phi$  and  $R_3$  from the other equations gives

$$(14.431) \quad A(A + ma^2)\ddot{\theta} + [C(C + ma^2)\omega_3^2 - Amga]\theta = \alpha,$$

where  $\alpha$  is a constant of integration. Obviously the condition for stability is

$$(14.432) \quad \omega_3^2 > \frac{Amga}{C(C + ma^2)}.$$

## 14.5. SUMMARY OF APPLICATIONS IN DYNAMICS IN SPACE— MOTION OF A RIGID BODY

### I. Rigid body with fixed point under no forces.

(a) For a general body, the motion is given by rolling the Poinot ellipsoid on the invariable plane; there is an analytic solution in terms of elliptic functions.

(b) For a body with an axis of symmetry, the Poinot ellipsoid is of revolution, and the motion is given by rolling the right circular body cone on the right circular space cone at a constant rate.

### II. The spinning top.

(a) Steady precession ( $p$ ) with fast spin ( $s$ ):

$$(14.501) \quad s = \frac{mga}{Cp} \quad (\text{approximately}).$$

(b) General motion expressible in terms of elliptic functions.

(c) Sleeping top stable if

(14.502)

$$s^2 > \frac{4Amga}{C^2}.$$

### III. Gyroscopes.

(a) A finite impulsive couple, applied to a fast-spinning gyroscope, alters the direction of the axis of rotation only slightly.

(b) Gyroscopic couple  $G$  required to maintain precession  $p$ :

(14.503)

$$G = Csp.$$

**Couple  $\rightarrow$  Precession  $\rightarrow$  Spin.**

(c) Gyrocompass:

(14.504)

$$\tau = 2\pi \sqrt{\frac{\Lambda}{Cs\Omega \cos \lambda}}.$$

### IV. General motion of a rigid body.

(a) Center of a slipping billiard ball describes a parabola.

(b) Rolling disk is stable if its angular velocity  $\omega$  satisfies

(14.505)

$$\omega^2 > \frac{\Lambda mga}{C(C + ma^2)}$$

### EXERCISES XIV

1. A circular disk, pivoted at its center, is set spinning with angular velocity  $\omega$  about a line making an angle  $\alpha$  with its axis. Find, in terms of  $\omega$  and  $\alpha$ , the time taken by the axis of the disk to describe a cone in space.

2. A gyroscope can turn freely about its mass center  $O$  which is fixed. Initially, it is set spinning about its axis, which is then struck perpendicularly. Find the angular velocity immediately after impact in terms of the initial spin, the moments of inertia, the magnitude of the impulsive force, and its distance from  $O$ . Draw a diagram showing the direction of the impulsive force, the angular velocity, and the angular momentum just after impact.

3. The center of a square plate is fixed. If at a certain instant the angular velocity vector makes an angle of  $30^\circ$  with the normal to the plate, find the inclination of the angular momentum vector to the normal.

4. A rigid body turns about a fixed point  $O$  under the action of a single force  $\mathbf{F}$  (in addition to the reaction at the fixed point). If the extremity of the angular momentum vector, drawn from  $O$ , lies in a fixed plane  $P$  throughout the motion, show that  $\mathbf{F}$  intersects

or is parallel to the perpendicular dropped from  $O$  on  $P$ .

5. A heavy homogeneous right circular cone spins with its vertex fixed. The axis of the cone is 4 in. long, and the radius of the base is 2 in. The axis maintains a constant inclination to the vertical and completes a rotation about the vertical in 5 sec. Find approximately the number of revolutions per second of the cone about its axis.

6. A lamina turns freely under no forces in three-dimensional motion about its mass center, which is fixed. Use Euler's equations to prove that the component of angular velocity in the plane of the lamina is constant in magnitude.

7. A rigid body turns about a fixed point under no forces. The momental ellipsoid at the fixed point is of revolution, and the axial moment of inertia  $C$  is greater than the transverse moment of inertia  $A$ . Show that if  $\alpha$  is the angle of inclination of the instantaneous axis to the axis of symmetry, then the semiangle of the space cone is

$$\tan^{-1} \frac{(C - A) \tan \alpha}{C + A \tan^2 \alpha}.$$

Noting the inequality  $C \leq A + B$ , satisfied in general by moments of inertia, show that the semiangle of the space cone cannot exceed

$$\tan^{-1} \frac{1}{\sqrt{2}}.$$

8. Make a rough estimate of the speed at which a twenty-five-cent piece must roll in order that its motion may be stable.

9. A solid homogeneous cuboid of edges  $2a$ ,  $2a$ ,  $4a$  can turn freely under no forces about its center, which is fixed. It is set spinning with angular velocity  $\omega$  about a diagonal. Find the semivertical angle of the cone described in space by the line through the center parallel to the longer edges, and show that the time taken by this line to move once round the cone is  $10\pi/(\omega \sqrt{11})$ .

10. A circular disk of mass  $m$  and radius  $a$  is made to roll without slipping in steady motion on a rough horizontal plane, its plane being vertical and its track a circle of radius  $b$ . It completes a circuit in time  $\tau$ . Reduce to a force at the center of the disk and a couple, the force system (including weight and the reaction of the plane) which must act on the disk in order that this motion may take place. The components of the force and the couple are to be expressed in terms of  $a$ ,  $b$ ,  $\tau$ ,  $m$ .

11. A rigid body with an axis of symmetry can turn about its mass center. There acts on it a frictional couple  $-\lambda\omega$ , where  $\omega$  is the angular velocity vector and  $\lambda$  a positive constant. Show that the axial component of angular velocity is reduced to half its original value in a time  $(C \log 2)/\lambda$ , where  $C$  is the axial moment of inertia. If  $C$  exceeds the transverse moment of inertia  $A$ , show also that the semiangle of the body cone decreases steadily.

12. An egg-shaped solid of revolution rolls in steady motion on a rough horizontal plane, the axis of figure being horizontal. Establish the relation

$$(a^2 + k^2)ns = abn^2 + bg,$$



where  $a$  is the radius of the greatest circular section,  $b$  the distance of the mass center from this section,  $k$  the radius of gyration about the axis of figure, and  $n, s$  the vertical and horizontal components of angular velocity.

13. Prove that a top cannot move in steady precession with spin  $s$  and inclination  $\theta$  to the vertical, unless

$$C^2 s^2 \geq 4A m g a \cos \theta.$$

14. A solid cone of height  $b$  and semivertical angle  $\alpha$  rolls in steady motion on a rough horizontal table, the line of contact rotating with angular velocity  $\Omega$ . Show that the reaction of the table on the cone is equipollent to a single force which cuts the generator of contact at a distance

$$\frac{3}{4} b \cos \alpha + \frac{k^2 \Omega^2}{g} \cot \alpha$$

from the vertex, where  $k$  is the radius of gyration of the cone about a generator. Deduce that the greatest possible value for  $\Omega$  is

$$\frac{1}{2k \cos \alpha} [g b \sin \alpha (1 + 3 \sin^2 \alpha)]^{\frac{1}{2}},$$

as the cone would overturn if  $\Omega$  exceeded this value.

15. A circular disk of radius  $a$  spins on a smooth table about a vertical diameter. Prove the motion is stable if the angular velocity exceeds  $2 \sqrt{g/a}$ .

16. A circular disk of radius  $a$  rolls on a rough horizontal plane in steady motion. The speed of its center is  $q_0$ , and its plane is inclined to the vertical at a constant angle  $\theta$ . Show that the radius  $r$  of the circle described by the center of the disk satisfies the equation

$$4gr^2 - 6q_0^2 r \cot \theta - q_0^2 a \cos \theta = 0;$$

deduce that, when  $\theta$  is small,

$$r = \frac{3q_0^2}{2g\theta}, \text{ approximately.}$$

17. A rigid body can turn freely about a smooth axis, for which its moment of inertia is  $I$ . It is acted on by a couple of constant magnitude  $G$ , applied in such a way that, when the body has turned through an angle  $\theta$ , the vector representing the couple makes an angle  $\theta$  with the axis. If the body is initially at rest, find its angular velocity when it has turned through a right angle.

If the axis is an axis of symmetry of the body, find the reaction exerted by the body on the axis when it has turned through an angle  $\theta$ .

18. A light axle  $L$  carries two gyroscopes;  $L$  is their common axis of symmetry, and

they can turn freely about it.  $L$  is so mounted that it can turn freely about a fixed point on it, halfway between the mass centers of the gyroscopes. Find a quadratic equation to determine the angular velocities of steady precession under the action of gravity, in terms of the following constants:

- $m, m'$ , the masses of the gyroscopes,
- $C, C'$ , their axial moments of inertia,
- $A, A'$ , their transverse moments of inertia at their mass centers,
- $s, s'$ , their spins,
- $2a$ , the distance between their centers,
- $\theta$ , the inclination of  $L$  to the vertical.

**19.** A rigid body turns about a fixed point under no forces. Show that, relative to the body, the extremity of the angular momentum vector  $\mathbf{h}$  moves on the curve of intersection of the sphere

$$x^2 + y^2 + z^2 = h^2$$

and the cone

$$x^2 \left( \frac{2T'}{h^2} - \frac{1}{A} \right) + y^2 \left( \frac{2T'}{h^2} - \frac{1}{B} \right) + z^2 \left( \frac{2T'}{h^2} - \frac{1}{C} \right) = 0,$$

the axes being principal axes of inertia.

Sketch the curves, taking  $A > B > C$  and considering all possible values of  $2T/h^2$ .

**20.** A top is spinning about its axis which is vertical. It is at the same time sliding over a smooth horizontal plane with velocity  $q_0$ . The vertex strikes a small smooth ridge on the plane, the direction of the motion being inclined at an angle  $\alpha$  to the direction of the ridge. If the coefficient of restitution for the impact is  $e$ , find the angle at which the vertex rebounds from the ridge in terms of  $q_0, \alpha, e$ , and the constants of the top. Find also the direction of motion of the mass center immediately after impact

**21.** A thin elliptical plate of semiaxes  $a, b$  ( $a > b$ ) can turn freely about its center, which is fixed; it is set in motion with an angular velocity  $\omega$  about an axis in its plane equally inclined to the axes of the ellipse. Show that the instantaneous axis will again be in the plane of the plate after a time

$$2 \int_0^\lambda (\lambda^4 - x^4)^{-1/2} dx,$$

where

$$\lambda^2 = \frac{n^2 a^2 - b^2}{2 a^2 + b^2}.$$

**22.** A thin hemispherical bowl of mass  $m$  and radius  $a$  stands on a smooth horizontal table. A horizontal impulsive force of magnitude  $\mathcal{P}$  is applied along a tangent to the rim.

becoming a luxury for which we cannot afford the time.

Instead of proceeding at once to Lagrange's equations in their full generality, we shall start with the case of a particle in a plane. Much of the difficulty of understanding the method may be overcome by a study of this comparatively simple case.

### **Lagrange's equations for a particle in a plane.**

Consider a particle of mass  $m$ , moving in a plane. Let  $Oxy$  be rectangular Cartesian axes, and let  $X, Y$  be the components of the force acting on the particle. The usual equations of motion are

$$(15.101) \quad m\ddot{x} = X, \quad m\ddot{y} = Y.$$

Now let  $q_1, q_2$  be any curvilinear coordinates (e.g., polar coordinates). It will be possible to express  $x$  and  $y$  in terms of  $q_1$  and  $q_2$  and so we may write

$$(15.102) \quad x = x(q_1, q_2), \quad y = y(q_1, q_2).$$

These relations hold for all values of the time  $t$ , and so

$$(15.103) \quad \dot{x} = \frac{\partial x}{\partial q_1} \dot{q}_1 + \frac{\partial x}{\partial q_2} \dot{q}_2, \quad \dot{y} = \frac{\partial y}{\partial q_1} \dot{q}_1 + \frac{\partial y}{\partial q_2} \dot{q}_2.$$

The partial derivatives occurring here are functions of  $q_1, q_2$ ; we can calculate them when the functions (15.102) are given.

Looking at (15.103) in a formal way and forgetting that  $\dot{q}_1$  is actually the derivative of  $q_1$ , we may regard them as equations expressing the two quantities  $\dot{x}, \dot{y}$  as functions of the four quantities  $q_1, q_2, \dot{q}_1, \dot{q}_2$ ; we may express this by writing

$$(15.104) \quad \dot{x} = f(q_1, q_2, \dot{q}_1, \dot{q}_2), \quad \dot{y} = g(q_1, q_2, \dot{q}_1, \dot{q}_2).$$

If we speak of the partial derivatives

$$\frac{\partial \dot{x}}{\partial q_1}, \quad \frac{\partial \dot{x}}{\partial q_2}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_1}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_2}$$

or the corresponding derivatives of  $\dot{y}$ , we understand that they are calculated from (15.104), all the quantities  $q_1, q_2, \dot{q}_1, \dot{q}_2$  being treated as constants, except the one with respect to which we differentiate.

But the functions in (15.104) are the same as those in (15.103), and so

$$(15.105) \quad \frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}, \quad \frac{\partial \dot{y}}{\partial \dot{q}_1} = \frac{\partial y}{\partial q_1}, \quad \frac{\partial \dot{y}}{\partial \dot{q}_2} = \frac{\partial y}{\partial q_2}.$$

Furthermore,  $\partial x / \partial q_1$  is a function of  $q_1, q_2$ ; so, following the motion of the particle,

$$(15.106) \quad \frac{d}{dt} \frac{\partial x}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2} \dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_1} \dot{q}_2.$$

But, on the other hand, if we differentiate the first of (15.103) partially with respect to  $q_1$ , we get

$$(15.107) \quad \frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2} \dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2} \dot{q}_2.$$

This is equal to the expression in (15.106). Hence, assembling this result with the other results obtained by using  $q_2$  and  $y$ , we have

$$(15.108) \quad \begin{cases} \frac{d}{dt} \frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial q_1}, & \frac{d}{dt} \frac{\partial y}{\partial q_1} = \frac{\partial \dot{y}}{\partial q_1}, \\ \frac{d}{dt} \frac{\partial x}{\partial q_2} = \frac{\partial \dot{x}}{\partial q_2}, & \frac{d}{dt} \frac{\partial y}{\partial q_2} = \frac{\partial \dot{y}}{\partial q_2}. \end{cases}$$

The equations (15.105) and (15.108) are fundamental in the development of Lagrange's equations.

The kinetic energy of the particle is

$$(15.109) \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2).$$

If we substitute for  $\dot{x}$  and  $\dot{y}$  from (15.103), we obtain a function of  $q_1, q_2, \dot{q}_1, \dot{q}_2$ ,

$$(15.110) \quad T = T(q_1, q_2, \dot{q}_1, \dot{q}_2).$$

Actually this function is of the form

$$(15.111) \quad T = \frac{1}{2}(a\dot{q}_1^2 + 2h\dot{q}_1\dot{q}_2 + b\dot{q}_2^2),$$

where  $a, h, b$  are functions of  $q_1, q_2$ .

Now, in (15.109),  $T$  is expressed as a function of  $\dot{x}, \dot{y}$ ; by (15.103),  $\dot{x}, \dot{y}$  are functions of  $q_1, q_2, \dot{q}_1, \dot{q}_2$ ; hence we obtain, using (15.105),

$$\begin{aligned}
 (15.112) \quad \frac{\partial T}{\partial \dot{q}_1} &= \frac{\partial T}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \dot{q}_1} + \frac{\partial T}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \dot{q}_1} \\
 &= m\dot{x} \frac{\partial x}{\partial q_1} + m\dot{y} \frac{\partial y}{\partial q_1}.
 \end{aligned}$$

Therefore, by (15.108) and (15.109), we have

$$(15.113) \quad \begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} = m\dot{x} \frac{\partial x}{\partial q_1} + m\dot{y} \frac{\partial y}{\partial q_1} + m\dot{x} \frac{\partial \dot{x}}{\partial q_1} + m\dot{y} \frac{\partial \dot{y}}{\partial q_1}, \\ \frac{\partial T}{\partial q_1} = m\dot{x} \frac{\partial x}{\partial q_1} + m\dot{y} \frac{\partial y}{\partial q_1}. \end{cases}$$

Subtracting and using the equations of motion (15.101), we get

$$(15.114) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = X \frac{\partial x}{\partial q_1} + Y \frac{\partial y}{\partial q_1}.$$

There is, of course, a similar equation with  $q_2$  instead of  $q_1$ .

Any small virtual displacement\* of the particle corresponds to increments  $\delta q_1, \delta q_2$  in the coordinates  $q_1, q_2$ . The corresponding increments in  $x, y$  are

$$(15.115) \quad \delta x = \frac{\partial x}{\partial q_1} \delta q_1 + \frac{\partial x}{\partial q_2} \delta q_2, \quad \delta y = \frac{\partial y}{\partial q_1} \delta q_1 + \frac{\partial y}{\partial q_2} \delta q_2.$$

The work done in this displacement is

$$(15.116) \quad \delta W = X \delta x + Y \delta y,$$

or

$$(15.117) \quad \delta W = Q_1 \delta q_1 + Q_2 \delta q_2,$$

where

$$(15.118) \quad Q_1 = X \frac{\partial x}{\partial q_1} + Y \frac{\partial y}{\partial q_1}, \quad Q_2 = X \frac{\partial x}{\partial q_2} + Y \frac{\partial y}{\partial q_2}.$$

Hence, we have the following result: *The motion of a particle in a plane satisfies the differential equations*

$$(15.119) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2,$$

where  $q_1, q_2$  are any curvilinear coordinates,  $T$  is the kinetic energy (expressed as a function of  $q_1, q_2, \dot{q}_1, \dot{q}_2$ ) and  $Q_1, Q_2$  are obtained from the expression  $\delta W$ , as in (15.117), for the work done in an arbitrary small displacement

These are *Lagrange's equations of motion*.

The curvilinear coordinates  $q_1, q_2$  are, of course, *generalized coordinates*, as discussed in Sec. 10.6; the quantities  $Q_1, Q_2$  are the *generalized forces* [cf. (10.708)].

It must be clearly understood that the Lagrangian method only provides the differential equations of motion; it does not *solve* them. It is true that the method does give some hints helpful for solution, but that is a matter into which we cannot go here.

*Example.* Consider a particle of mass  $m$  moving in a plane, under an attractive force  $\mu m/r^2$ , directed to the origin of polar coordinates  $r, \theta$ . If we take as generalized coordinates

$$q_1 = r, q_2 = \theta,$$

and denote the generalized forces by  $R, \Theta$ , the equations of motion (15.119) read

$$(15.120) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = R, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = \Theta.$$

Now

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and so

$$(15.121) \quad \frac{\partial T}{\partial r} = m\dot{\theta}^2, \quad \frac{\partial T}{\partial \dot{r}} = mr\dot{\theta}, \quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0.$$

To find  $R$  and  $\Theta$ , we have (for an arbitrary displacement  $\delta r, \delta \theta$ )

$$R \delta r + \Theta \delta \theta = \delta W = -\frac{\mu m}{r^2} \delta r,$$

and so

$$(15.122) \quad R = -\frac{\mu m}{r^2}, \quad \Theta = 0.$$

Substituting from (15.121) and (15.122) in (15.120), we get

$$(15.123) \quad m\ddot{r} - mr\dot{\theta}^2 = -\frac{\mu m}{r^2}, \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0.$$

These equations are the same as (5.104); the present method of obtaining them is simpler than the method used earlier.

## 15.2. LAGRANGE'S EQUATIONS FOR A GENERAL SYSTEM

## Lagrange's equations for a system with two degrees of freedom.

We pass now from a particle moving in a plane to any system with two degrees of freedom, with generalized coordinates  $q_1, q_2$  (cf. Sec. 10.6). Let  $N$  be the number of particles forming the system, and let the Cartesian coordinates of a particle (of mass  $m_i$ ) be  $x_i, y_i, z_i$  ( $i = 1, 2, \dots, N$ ). Then  $x_i, y_i, z_i$  are functions of  $q_1, q_2$ , and we may write

$$(15.201) \quad x_i = x_i(q_1, q_2), \quad y_i = y_i(q_1, q_2), \quad z_i = z_i(q_1, q_2).$$

Here we have  $3N$  equations like the two equations (15.102), and we obtain on differentiation  $3N$  equations like (15.103),

$$(15.202) \quad \begin{cases} \dot{x}_i = \frac{\partial x_i}{\partial q_1} \dot{q}_1 + \frac{\partial x_i}{\partial q_2} \dot{q}_2, \\ \dot{y}_i = \frac{\partial y_i}{\partial q_1} \dot{q}_1 + \frac{\partial y_i}{\partial q_2} \dot{q}_2, \\ \dot{z}_i = \frac{\partial z_i}{\partial q_1} \dot{q}_1 + \frac{\partial z_i}{\partial q_2} \dot{q}_2. \end{cases}$$

As a matter of fact, the whole argument for a system with two degrees of freedom follows very closely the argument for a particle in a plane; the complication introduced by having  $3N$  Cartesian coordinates, instead of only two, is not serious. Thus, if we use a symbol  $\xi$  to stand for any one of the coordinates  $x_i, y_i, z_i$ , we obtain, exactly as in (15.105) and (15.108), the following equations:

$$(15.203) \quad \frac{\partial \xi}{\partial \dot{q}_1} = \frac{\partial \xi}{\partial \dot{q}_1}, \quad \frac{\partial \xi}{\partial \dot{q}_2} = \frac{\partial \xi}{\partial \dot{q}_2},$$

$$(15.204) \quad \frac{d}{dt} \frac{\partial \xi}{\partial \dot{q}_1} = \frac{\partial \xi}{\partial q_1}, \quad \frac{d}{dt} \frac{\partial \xi}{\partial \dot{q}_2} = \frac{\partial \xi}{\partial q_2}.$$

The kinetic energy of the system is

$$(15.205) \quad T = \frac{1}{2} \sum_{i=1}^N m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2),$$

and this is expressible in the form

$$(15.206) \quad T = T(q_1, q_2, \dot{q}_1, \dot{q}_2).$$

As in the case of the single particle, this is a quadratic expression



with two degrees of freedom, of which a particle in a plane is, of course, a special case.

When the system is conservative, with potential energy  $V(q_1, q_2)$ , the generalized forces are connected with  $V$  by (10.71.2). Thus, (15.213) may be written

$$(15.214) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = - \frac{\partial V}{\partial q_1}, \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = - \frac{\partial V}{\partial q_2}.$$

*Example 1.* We shall now find the equations of motion of a spherical pendulum. Let  $m$  be the mass of the particle and  $a$  the radius of the sphere on which the particle is constrained to move. We take as generalized coordinates

$$q_1 = \theta, \quad q_2 = \phi,$$

where  $\theta$  is the angular distance from the highest point of the sphere and  $\phi$  the azimuthal angle. Then,

$$T = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2), \quad V = mga \cos \theta,$$

and so

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= ma^2 \dot{\theta}, & \frac{\partial T}{\partial \phi} &= ma^2 \sin \theta \cos \theta \dot{\phi}, & \frac{\partial V}{\partial \theta} &= -mga \sin \theta, \\ \frac{\partial T}{\partial \dot{\theta}} &= ma^2 \dot{\theta}, & \frac{\partial T}{\partial \dot{\phi}} &= 0, & \frac{\partial V}{\partial \dot{\phi}} &= 0. \end{aligned}$$

Thus (15.214) give, as equations of motions of a spherical pendulum,

$$\begin{aligned} ma^2 \ddot{\theta} - ma^2 \sin \theta \cos \theta \dot{\phi}^2 &= mga \sin \theta, \\ \frac{d}{dt} (ma^2 \sin^2 \theta \dot{\phi}) &= 0. \end{aligned}$$

*Example 2.* Consider a uniform bar hanging by one end from a smooth horizontal rail. It can move only in the vertical plane through the rail and is under the influence of gravity and a horizontal force  $X$  applied to its lowest point. Let us find the equations of motion.

For generalized coordinates, we take

$q_1$  = distance of point of suspension from some fixed point on rail,

$q_2$  = inclination of bar to vertical.

Then,

$$T = \frac{1}{2}m \left\{ \left[ \frac{d}{dt} (q_1 + a \sin q_2) \right]^2 + \left[ \frac{d}{dt} (a \cos q_2) \right]^2 \right\} + \frac{1}{2}mk^2 \dot{q}_2^2,$$

where  $m$  = mass of bar,

$2a$  = length of bar,

$k$  = radius of gyration about mass center.

The above expression reduces to

$$T = \frac{1}{2}m[\dot{q}_1^2 + 2a \cos q_2 \dot{q}_1 \dot{q}_2 + (a^2 + k^2)\dot{q}_2^2].$$

The generalized forces are given by

$$Q_1 \delta q_1 + Q_2 \delta q_2 = X \delta(q_1 + 2a \sin q_2) + mg \delta(a \cos q_2).$$

Thus

$$Q_1 = X, \quad Q_2 = 2Xa \cos q_2 - mga \sin q_2,$$

and so the equations of motion are, by (15.213),

$$m \frac{d}{dt} (\dot{q}_1 + a \cos q_2 \dot{q}_2) = X,$$

$$m \frac{d}{dt} [a \cos q_2 \dot{q}_1 + (a^2 + k^2)\dot{q}_2] + ma \sin q_2 \dot{q}_1 \dot{q}_2 = 2Xa \cos q_2 - mga \sin q_2.$$

If the bar remains nearly vertical, so that  $q_2$  is small, these equations simplify to the approximate form

$$\begin{aligned} q_1 + a q_2 &= \frac{X}{m}, \\ \dot{q}_1 + \frac{1}{2}a \dot{q}_2 + g q_2 &= \frac{2X}{m}. \end{aligned}$$

## Lagrange's equations for a general system.

Consider a system with  $n$  degrees of freedom and generalized coordinates  $q_1, q_2, \dots, q_n$ . The method of finding Lagrange's equations in this general case differs from the method given above only in a slightly greater complexity, due to the  $n$  degrees of freedom. We shall give here an argument complete in essentials but omitting details which can be supplied by the type of argument used earlier.\*

Let  $m_i, x_i, y_i, z_i$  ( $i = 1, 2, \dots, N$ ) be the mass and coordinates of the  $i$ th particle. Then, for  $r = 1, 2, \dots, n$ ,

motion, it may sometimes be used indirectly to give information not easy to obtain otherwise. Consider a particle moving in space. Let us take the spherical polar coordinates  $r, \theta, \phi$  as generalized coordinates  $q_1, q_2, q_3$ . Then, if the particle is of unit mass,

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2).$$

(To obtain this, we need only the components of velocity along the parametric lines.) If  $R, \Theta, \Phi$  are the generalized forces, the equations of motion are, by (15.215),

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 &= R, \\ \frac{d}{dt}(r^2\dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 &= \Theta, \\ \frac{d}{dt}(r^2 \sin^2 \theta \dot{\phi}) &= \Phi. \end{aligned}$$

Let  $f_r, f_\theta, f_\phi$  be the components of acceleration along the parametric lines. These are equal to the components of force in these directions. Hence, equating two different expressions for work done in an arbitrary displacement, we have

$$\delta W = f_r \delta r + f_\theta \delta \theta + f_\phi r \sin \theta \delta \phi = R \delta r + \Theta \delta \theta + \Phi \delta \phi,$$

and so

$$(15.301) \quad \begin{cases} f_r = R = \ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2, \\ f_\theta = \frac{1}{r} \Theta = \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) - r \sin \theta \cos \theta \dot{\phi}^2, \\ f_\phi = \frac{1}{r \sin \theta} \Phi = \frac{1}{r \sin \theta} \frac{d}{dt}(r^2 \sin^2 \theta \dot{\phi}). \end{cases}$$

These are the components of acceleration along the parametric lines of spherical polar coordinates.

### Normal frequencies of vibration of a system with two degrees of freedom.

Let  $C$  be a position of equilibrium of a conservative system with two degrees of freedom. Let us choose generalized coordinates such that  $q_1 = q_2 = 0$  at  $C$ . The kinetic energy is expressible in the form

$$(15.302) \quad T = \frac{1}{2}(a\dot{q}_1^2 + 2h\dot{q}_1\dot{q}_2 + b\dot{q}_2^2),$$

where  $a, h, b$  are functions of  $q_1, q_2$ . Let us, however, consider only *small*

oscillations about  $C$ , so that  $q_1, q_2, \dot{q}_1, \dot{q}_2$  are small. Then the principal part of  $T$  has the form (15.302), where  $a, h, b$  are constants, viz., the values of the coefficients for  $q_1 = q_2 = 0$ .

Consider now the potential energy  $V$ . We may choose  $C$  as standard configuration, so that  $V = 0$  for  $q_1 = q_2 = 0$ . The expansion of  $V$  in a Taylor series reads

$$(15.303) \quad V = \frac{\partial V}{\partial q_1} q_1 + \frac{\partial V}{\partial q_2} q_2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial q_1^2} q_1^2 + 2 \frac{\partial^2 V}{\partial q_1 \partial q_2} q_1 q_2 + \frac{\partial^2 V}{\partial q_2^2} q_2^2 \right) + \dots,$$

where the partial derivatives are evaluated  $q_1 = q_2 = 0$ . But, by the principle of virtual work (10.714),

$$\frac{\partial V}{\partial q_1} = 0, \quad \frac{\partial V}{\partial q_2} = 0,$$

for  $q_1 = q_2 = 0$ . Hence the principal part of  $V$  is

$$(15.304) \quad V = \frac{1}{2}(Aq_1^2 + 2Hq_1q_2 + Bq_2^2),$$

where  $A, H, B$  are constants. Thus, to our approximation,  $T$  and  $V$  are homogeneous quadratic forms with constant coefficients,  $T$  being quadratic in the velocities and  $V$  in the coordinates.

We have

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_1} &= a\dot{q}_1 + h\dot{q}_2, & \frac{\partial T}{\partial q_1} &= 0, & \frac{\partial V}{\partial q_1} &= Aq_1 + Hq_2, \\ \frac{\partial T}{\partial \dot{q}_2} &= h\dot{q}_1 + b\dot{q}_2, & \frac{\partial T}{\partial q_2} &= 0, & \frac{\partial V}{\partial q_2} &= Hq_1 + Bq_2, \end{aligned}$$

and so Lagrange's equations read

$$(15.305) \quad \begin{cases} a\ddot{q}_1 + h\ddot{q}_2 = -(Aq_1 + Hq_2), \\ h\ddot{q}_1 + b\ddot{q}_2 = -(Hq_1 + Bq_2). \end{cases}$$

We seek a solution of the form

$$q_1 = \alpha \cos (nt + \epsilon), \quad q_2 = \beta \cos (nt + \epsilon).$$

When we substitute in (15.305) and eliminate  $\alpha$  and  $\beta$ , we obtain for  $n$  the determinantal equation

momentum.) When we substitute in the first of (15.309), we get a differential equation for  $\theta$

$$(15.311) \quad \ddot{\theta} + \frac{(\alpha - \beta \cos \theta)(\beta - \alpha \cos \theta)}{A^2 \sin^3 \theta} = \frac{mga}{A} \sin \theta.$$

If we multiply this equation by  $\dot{\theta}$ , integrate once, and put  $\cos \theta = x$ , we get (14.226). The detailed theory of the motion then proceeds as in Sec. 14.2.

### Lagrange's equations for impulsive forces.

When impulsive forces act, there are instantaneous changes in velocity, without instantaneous changes in position. In terms of generalized coordinates  $q_r$ , there are instantaneous changes in  $\dot{q}_r$ , but not in  $q_r$ . As usual, we approach impulsive forces by a limiting process, in which the forces tend to infinity and the interval during which they act tends to zero. We multiply (15.215) by  $dt$  and integrate over the interval  $(t_0, t_1)$ . When  $t_1 \rightarrow t_0$ , the second term on the left disappears, and we have *Lagrange's equations for impulsive forces*,

$$(15.312) \quad \Delta \frac{\partial T}{\partial \dot{q}_r} = \hat{Q}_r, \quad (r = 1, 2, \dots, n);$$

here  $\Delta$  denotes a sudden increment and  $\hat{Q}_r$  are the *generalized impulsive forces* [cf. (8.112)]

$$(15.313) \quad \hat{Q}_r = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} Q_r dt.$$

These may be calculated from a formula analogous to (15.216),

$$(15.314) \quad \delta \hat{W} = \sum_{r=1}^n \hat{Q}_r \delta q_r,$$

where  $\delta \hat{W}$  is the work which would be done in a general displacement by the impulsive forces if they were ordinary forces.

The Lagrangian method is particularly useful for systems of linked rods, because the impulsive reactions are automatically eliminated. Thus, to take an example, consider the problem worked in Sec. 8.3 (Figs. 99a and 99b). As generalized coordinates we take  $x, y$ , the coordinates of the joint and  $\theta_1, \theta_2$ , the inclinations of the rods to their initial line. Then, for the given position ( $\theta_1 = \theta_2 = 0$ ),

$$T = \frac{1}{2}m[\dot{x}^2 + (\dot{y} - a\dot{\theta}_1)^2 + k^2\dot{\theta}_1^2 + \dot{x}^2 + (\dot{y} + a\dot{\theta}_2)^2 + k^2\dot{\theta}_2^2],$$

where  $k$  is the radius of gyration of a rod about its center. Now if  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{\theta}_1$ ,  $\hat{\theta}_2$  are the generalized impulsive forces, we have

$$\hat{X} \delta x + \hat{Y} \delta y + \hat{\theta}_1 \delta \theta_1 + \hat{\theta}_2 \delta \theta_2 = \hat{P}(\delta y + 2a \delta \theta_2).$$

Thus

$$\hat{X} = 0, \quad \hat{Y} = \hat{P}, \quad \hat{\theta}_1 = 0, \quad \hat{\theta}_2 = 2a\hat{P}.$$

Lagrange's equations give

$$\begin{aligned} 2m\dot{x} &= 0, \\ m(\dot{y} - a\dot{\theta}_1) + m(\dot{y} + a\dot{\theta}_2) &= \hat{P}, \\ -ma(\dot{y} - a\dot{\theta}_1) + mk^2\dot{\theta}_1 &= 0, \\ ma(\dot{y} + a\dot{\theta}_2) + mk^2\dot{\theta}_2 &= 2a\hat{P}. \end{aligned}$$

Hence we obtain, with  $k^2 = a^2/3$ ,

$$\dot{x} = 0, \quad \dot{y} = -\frac{\hat{P}}{m}, \quad \dot{\theta}_1 = -\frac{3}{4}\frac{\hat{P}}{ma}, \quad \dot{\theta}_2 = \frac{3}{4}\frac{\hat{P}}{ma}.$$

## 15.4. SUMMARY OF LAGRANGE'S EQUATIONS

### I. Finite forces.

For system with kinetic energy  $T$ , expressed as function of  $q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ :

$$(15.401) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = Q_r, \quad (r = 1, 2, \dots, n).$$

$$(15.402) \quad \delta W = \sum_{r=1}^n Q_r \delta q_r.$$

$$(15.403) \quad Q_r = -\frac{\partial V}{\partial q_r}, \quad \text{for conservative system.}$$

### II. Impulsive forces.

$$(15.404) \quad \Delta \frac{\partial T}{\partial \dot{q}_r} = \hat{Q}_r, \quad (r = 1, 2, \dots, n).$$

$$(15.405) \quad \delta \hat{W} = \sum_{r=1}^n \hat{Q}_r \delta q_r.$$

## EXERCISES XV

(To be done by Lagrange's equations)

1. Find the equation of motion of a simple pendulum, taking in turn the following generalized coordinates:

- (i) the angular displacement,
- (ii) the horizontal displacement,
- (iii) the vertical displacement.

2. Find the equation of motion of a sphere rolling down a rough inclined plane.

3. Find the equations of motion of a spherical pendulum, taking as generalized coordinates the horizontal Cartesian coordinates of the bob. Reduce the equations to their principal parts for oscillations near the equilibrium position.

4. Four flywheels with moments of inertia  $I_1, I_2, I_3, I_4$  are connected by light gearing so that their angular velocities are in fixed ratios  $n_1:n_2:n_3:n_4$ . Driving torques  $N_1, N_2, N_3, N_4$  are applied to the flywheels. Find their angular accelerations.

5. Show that if a generalized coordinate ( $q_1$ ) does not appear explicitly in either  $T$  or  $V$ , then  $\partial T / \partial \dot{q}_1$  is constant throughout the motion.

A rod hangs by a universal joint from its upper end. For oscillations under gravity, use the above result and the equation of energy to find a differential equation of the first order for  $\theta$ , the inclination of the rod to the vertical.

6. A pendulum consists of two equal bars  $AB, BC$ , smoothly jointed at  $B$  and suspended from  $A$ . The mass of each bar is  $m$ , and its length is  $2a$ . Find the normal periods for small oscillations in a vertical plane under gravity, in the form

$$2\pi\lambda \sqrt{\frac{a}{g}}$$

where  $\lambda$  is a numerical constant.

7. The pendulum described in Exercise 6 hangs at rest. A horizontal impulse  $\mathbf{P}$  is applied at its lowest point. Find the angular velocities imparted to the bars.

8. The ends of a heavy uniform bar of mass 120 lb. are supported by springs of equal strength, the bar being horizontal. The strength of the springs is such that a weight  $W$  of 100 lb, placed gently at the middle point of the bar, causes it to descend 1 in. Find, to two significant figures, the normal frequencies of small vibrations of the bar (without the weight  $W$ ), considering only vibrations in which the springs move vertically

9. On a sphere,  $\theta$  and  $\phi$  are polar angles. A particle describes a small circle  $\theta = \text{constant}$  with constant speed  $q_0$ . Find the generalized forces  $\Theta, \Phi$  consistent with this motion.

10. Consider a dynamical system with kinetic and potential energies

$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \quad V = f(q_1 - q_2),$$



where  $f$  is a given function. By choosing suitable new coordinates  $q'_1, q'_2$ , reduce the problem of determining the motion to the evaluation of an integral involving the function  $f$ . Determine  $q_1, q_2$  as functions of  $t$  if  $f(x) = x^2$ .

11. A carriage has four wheels, each of which is a uniform disk of mass  $m$ . The mass of the carriage without the wheels is  $M$ . The carriage rolls without slipping down a plane slope inclined to the horizontal at an angle  $\alpha$ , the floor of the carriage remaining parallel to the slope. A perfectly rough spherical ball of mass  $m'$  rolls on the floor of the carriage along a line parallel to a line of greatest slope. Show that the acceleration of the carriage down the plane is

$$\frac{7M + 28m + 2m'}{7M + 42m + 2m'} \cdot g \sin \alpha,$$

and find the acceleration of the ball.

12. A rhombus of equal rods, smoothly jointed, lies on a plane in the form of a square. An impulse is applied to one corner, along the diagonal through that corner. Find the angular velocities imparted to the rods, in terms of the impulse ( $\hat{p}$ ), the mass ( $m$ ) of a rod, and the length ( $2a$ ) of a rod.

13. A smooth circular wire carries a bead. The wire is suspended from a point on it. Find the normal periods of small vibrations under gravity when the wire swings in its own plane and the bead slides on the wire. Show that, when the bead is fixed to the wire at its position of equilibrium when free to slide, the period coincides with one of these two normal periods.

14. Using the fact that  $T$  is a homogeneous quadratic expression in the generalized velocities  $\dot{q}_r$ , show that the integral of energy  $T + V = \text{constant}$  may be proved as a mathematical deduction from Lagrange's equations. Note that, if  $f$  is a homogeneous function of degree  $m$  in  $x_1, x_2, \dots, x_n$ , then

$$mf = \sum_{r=1}^n x_r \frac{\partial f}{\partial x_r}.$$

15. A dynamical system has kinetic energy

$$T = \frac{1}{2}(a\dot{q}_1^2 + 2h\dot{q}_1\dot{q}_2 + b\dot{q}_2^2),$$

and potential energy

$$V = \frac{1}{2}(Aq_1^2 + 2Hq_1q_2 + Bq_2^2).$$

Additional generalized forces

$$Q_1 = -k_{11}\dot{q}_1 - k_{12}\dot{q}_2, \quad Q_2 = -k_{21}\dot{q}_1 - k_{22}\dot{q}_2$$

are applied. All the coefficients  $a, h, b, A, H, B$ , and the  $k$ 's are constants. Show that the energy sum  $T + V$  decreases steadily during any motion, provided

$$k_{11} > 0, \quad 4k_{11}k_{22} > (k_{12} + k_{21})^2.$$

**16.** A gyroscope is mounted in a light Cardan's suspension (Fig. 144). Take Eulerian angles simply related to the suspension, and find the equations of motion of the system under the action of a couple  $\mathbf{G}$  applied to the outer ring,  $\mathbf{G}$  being in the line of the outer bearings.

**17.** A system is said to have "moving constraints" when the configuration of the system is determined by the values of generalized coordinates  $q_1, q_2, \dots, q_n$  and the value of the time  $t$ . Show that, for such a system, Lagrange's equations hold in the same form as when there are no moving constraints, but that the kinetic energy is no longer a homogeneous quadratic expression in  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ .

Apply this result to find the equation of motion of a heavy bead on a smooth circular wire, the wire being made to rotate about the vertical diameter with constant angular velocity.

\* It should be emphasized that this virtual displacement is arbitrary; it is not to be confused with the displacement actually occurring in the motion. If we want to refer to the latter, we write  $dx, dy, dq_1, dq_2$ .

\* As in Sec. 10.6, non-holonomic systems will not be considered.

\* Except where forces of friction do work.

in which the clock was treated on the flight—whether it was knocked about, whether it was subjected to extremes of heat and cold, and so on. Let us suppose that the greatest care has been taken, so that effects due to these accidental causes may be ruled out of consideration. Then the answers are as follows:

Newtonian theory: The clocks will show the same reading.

Relativity theory: The readings will not be the same. The clock that has been on the flight will be slow in comparison with the clock that has stayed at home.

The follower of Newton reasons along these lines: A perfect clock registers *the* time. A flight in an airplane does not alter this fact, provided that proper precautions are taken. Since after the flight each clock registers *the* time, they must agree.

We cannot yet give the reasoning of the relativist; that will come later in the chapter. For the present, we must be satisfied with the words with which the relativist would begin his attack on the argument of the Newtonian: There is no such thing as *the* time, in any absolute sense.

It would be impossible to decide between the two predictions by carrying out the experiment we have described. The relativist would predict a difference between the two readings far too small to detect. The airplane would have to fly with a speed comparable with that of light before the effect would be noticeable. But it is the *principle* that is important. Other experiments can be carried out in which the predictions of the two theories are different and the difference is large enough to measure; in every case the relativistic prediction proves correct. There can be no doubt that the theory of relativity gives us a mathematical model closer to nature than the Newtonian model. We must therefore pay attention to the words: There is no such thing as *the* time, in any absolute sense. Once that point is conceded, the basis of the Newtonian pattern is broken, and the way is open for relativity.

## **The ingredients of relativity.**

The theory of relativity is divided into two parts:

- (i) the special theory;
- (ii) the general theory.

The special theory deals with phenomena in which gravitational attraction plays no part, while the general theory might be called “Einstein’s theory of gravitation.” In this book, we shall be concerned solely with the special theory.

At this stage the reader should glance over [Chap. I](#) to concentrate his

We recall that, in [Chap. I](#), we introduced the idea of an *event*—something happening suddenly at a point. We carry this idea over into relativity, where we shall make extensive use of it. Even though his clocks are not yet synchronized, the observer is prepared to describe any event by assigning four coordinates to it. Of these coordinates, three are spatial ( $x, y, z$ ), and the fourth ( $t$ ) is given by the local clock, i.e., the clock situated at the point where the event occurs.

## **Galilean frames of reference.**

We have already seen in Newtonian mechanics the importance of making a proper choice of frame of reference. The laws of Newtonian mechanics take their simplest form only in certain special frames, which we called Newtonian. Similarly, in relativity there are frames of reference which are particularly convenient to use. These are called *Galilean frames of reference*.<sup>\*</sup> They correspond in nature to rigid bodies situated in remote space, far from attracting matter, and without rotation relative to the stars as a whole.

We shall now make the following hypothesis regarding a Galilean frame of reference:

I. *A Galilean frame of reference is a rigid body, isotropic with respect to mechanical and optical experiments.*

To explain this, we note that “isotropic” means “the same in all directions.” The neighborhood of the earth is not isotropic. If we drop a stone, it falls in a definite direction and the earth’s rotation defines a direction which we can detect by means of a gyrocompass. However, in applying the theory of relativity, we may often regard the earth as a Galilean frame, for its gravitational attraction may be small compared with other forces involved and its rotation may be of no importance.

The assumption—that a rigid body in remote space is isotropic—is at least plausible. To assert that it was not isotropic would at once raise the question: Why should any one direction be privileged above another?

The hypothesis refers to mechanical and optical experiments. We must provide the observer with apparatus to perform these. We shall therefore give him mechanisms by which he can exert *forces*, and lamps and mirrors by which he can send out *flashes of light* and reflect them.

In Newtonian mechanics, we had no occasion to refer to light. Optics appeared to be a separate subject. In relativity, on the other hand, we have to discuss optics and mechanics together.

## **The synchronization of clocks.**



cannot be exceeded is something essentially new.

To sum up, we have made three hypotheses in all. The first deals with a single Galilean frame of reference; the last two concern the relations between two Galilean frames of reference.

## The Lorentz transformation.

Let  $S$  and  $S'$  be two Galilean frames of reference. (We may without confusion also use these letters for observers in the two frames.) Consider any event, observed by both observers. To this event,  $S$  attaches coordinates  $(x, y, z, t)$ , and  $S'$  attaches coordinates  $(x', y', z', t')$ . In doing this, each observer uses his own measuring rod and clocks. The event determines the coordinates, and conversely the coordinates determine the event. Thus, considering all possible events, four numbers  $(x, y, z, t)$  determine an event, and that in turn determines the four numbers  $(x', y', z', t')$ . The last four numbers are therefore functions of the first four, and we express this by writing

$$(16.201) \quad \begin{aligned} x' &= f(x, y, z, t), & y' &= g(x, y, z, t), \\ z' &= h(x, y, z, t), & t' &= l(x, y, z, t). \end{aligned}$$

Such relations, connecting the coordinates of two Galilean observers, constitute a *Lorentz transformation*.

We have now to investigate the forms of these functions. We shall not, however, suppose that the axes  $Oxyz$  and  $O'x'y'z'$  are given arbitrary directions in the respective frames. We shall suppose them so chosen that  $Ox$  and  $O'x'$  lie on a common line when viewed by either observer, this line being parallel to the relative velocity of either frame with respect to the other. We shall consider the Lorentz transformation only for events occurring on this common line. Thus  $y = z = y' = z' = 0$ , and the transformation is of the form

$$(16.202) \quad x' = f(x, t), \quad t' = l(x, t).$$

We must carefully avoid the idea that there is any “absolute” frame from which  $S$  and  $S'$  may be viewed. We must look at things either as they appear to  $S$  or as they appear to  $S'$ . First,  $S$  sees the particles of his own frame. They are fixed as far as he is concerned, and through them there pass the particles of the frame  $S'$ . These particles all move parallel to  $Ox$  with a constant speed  $V$ , the speed of  $S'$  relative to  $S$ . Similarly, to  $S'$  the particles of *his* frame appear fixed, and the particles of  $S$  pass with a speed  $V'$ , directed in the negative sense of  $O'x'$ . To do justice to both observers, it is best to draw two diagrams, as in [Figs. 159a](#) and [b](#). In [Fig. 159a](#) we take the view of  $S$  and regard  $Oxyz$  as stationary; in [Fig. 159b](#) we take the view of  $S'$  and regard  $O'x'y'z'$  as stationary.

The two observers now fix their attention on a flash of light traveling along the common line  $Ox, O'x'$ . It will add to the complexity of our work if we assume that the units of space and time used by  $S$  and  $S'$  are completely independent. We shall therefore suppose that they have been supplied with measuring rods and clocks from a common stock. Then, by virtue of the principle of equivalence, the speed of light\* has a common value in the two frames. This common value we shall denote by  $c$ .

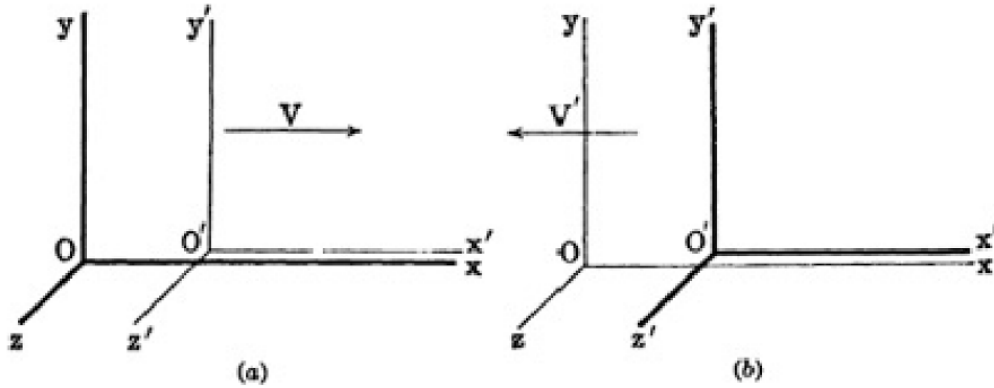


FIG. 159.—(a) The frame  $S'$  moving relative to the frame  $S$ . (b) The frame  $S$  moving relative to the frame  $S'$ .

As  $S$  observes the flash, he records the time  $t$  at which (by his local clock) the flash reaches the position  $x$ . Since the speed of light is  $c$ ,  $x$  is a function of  $t$  satisfying

$$\left(\frac{dx}{dt}\right)^2 = c^2.$$

But similarly, for the observations of  $S'$ ,

$$\left(\frac{dx'}{dt'}\right)^2 = c^2.$$

Thus, for the sequence of events given by the passage of the flash, we have the two equations

$$dt^2 - dx^2/c^2 = 0, \quad dt'^2 - dx'^2/c^2 = 0.$$

Now a motion satisfying either of these equations represents the passage of a flash of light and therefore must satisfy the other equation. Thus, if one of the equations is satisfied, so is the other, and therefore we have the identity

$$(16.203) \quad dt'^2 - dx'^2/c^2 \equiv k (dt^2 - dx^2/c^2),$$

where  $k$  is some unknown factor. But by the principle of equivalence we must

To a person accustomed to thinking in the Newtonian way, some of the predictions of the theory of relativity are startling. Outstanding among these are the contraction of a moving body and the slowing down of a moving clock. These apparently curious facts are consequences of the Lorentz transformation (16.213).

First, let us consider a measuring rod which  $S'$  lays down along his axis  $Ox'$ . To him it is a fixed measuring rod. If  $A, B$  are its ends, the history of  $A$  is a sequence of events for which  $x' = (x')_A$ , a constant, and the history of  $B$  is a sequence of events for which  $x' = (x')_B$ , also a constant; the length of the rod is

$$(16.216) \quad L' = (x')_B - (x')_A.$$

Viewed by  $S$ , the rod is not fixed. An instantaneous picture, taken by  $S$  at time  $t$ , shows  $A$  at  $(x)_A$ , say, and  $B$  at  $(x)_B$ . If asked what is the *apparent* length of the rod,  $S$  naturally says that it is

$$(16.217) \quad L = (x)_B - (x)_A.$$

Now, by the first equation of (16.213),

$$(16.218) \quad \begin{cases} (x')_B = \gamma[(x)_B - Vt], \\ (x')_A = \gamma[(x)_A - Vt], \end{cases}$$

and subtraction gives, in view of (16.216) and (16.217),

$$L' = \gamma L,$$

or

$$(16.219) \quad L = L'/\gamma = L' \sqrt{1 - V^2/c^2} < L'.$$

Thus  $L$  is less than  $L'$ ; the rod appears to  $S$  to be contracted in the ratio  $\sqrt{1 - V^2/c^2}:1$ .

We might expect that, if  $S'$  viewed a rod fixed in  $S$ , he would see an expansion instead of a contraction. But if we carry out the calculation, now using (16.214) instead of (16.213), we find the same contraction again. *Each observer considers that the measuring rod of the other is contracted.*

Now let us consider a clock carried along in  $S'$ . Let  $(t')_A, (t')_B$  be two readings of the clock. These are two events, both with the same  $x'$ , and with  $t' = (t')_A, t' = (t')_B$ , respectively. Viewed by  $S$ , the clock is moving; let the times of the two events be  $(t)_A, (t)_B$ , respectively, as measured in his time system. From



the second equation of (16.214), we have

$$(16.220) \quad \begin{cases} (t)_B = \gamma \left[ (t')_B + \frac{Vx'}{c^2} \right], \\ (t)_A = \gamma \left[ (t')_A + \frac{Vx'}{c^2} \right]. \end{cases}$$

By subtraction,

$$(t)_B - t_{(A)} = \gamma[(t')_B - (t')_A],$$

or

$$(16.221) \quad T = \gamma T' = \frac{T'}{\sqrt{1 - V^2/c^2}} > T',$$

where  $T$  is the time interval recorded by  $S$  and  $T'$  the time interval recorded by  $S'$ . Since  $T' < T$ , the clock carried by  $S'$  appears to  $S$  to be running slow. Just as in the case of contraction of length, this result works both ways. *Each observer considers the clock of the other to be running slow.*

## Space-time.

It does not seem possible at first sight to do justice simultaneously to each of two Galilean observers, for it appears necessary to take the point of view of either the one or the other. This difficulty is overcome by using a *space-time diagram*.

There is nothing peculiarly relativistic about a space-time diagram. We have used the idea in Newtonian mechanics, as in Fig. 78, when we plotted the position of a damped harmonic oscillator against the time. But it is in relativity that we get full advantage from this idea.

Consider first one Galilean observer  $S$ . Draw oblique Cartesian axes on a sheet of paper, and label them  $\Omega x$ ,  $\Omega t$  (Fig. 160). (We use  $\Omega$  instead of  $O$  for origin, to avoid confusion with the origin of the observer's axes.) Any event which happens on the axis of  $Ox$  in the observer's frame will have attached to it values of  $x$  and  $t$ . It can then be represented by a point in Fig. 160, which is our space-time diagram. The history of a particle moving along  $Ox$  will appear as a curve  $C$  in the space-time diagram. If it moves with uniform velocity,  $dx/dt$  is a constant, and the curve becomes a straight line  $C_1$ .

Consider now a second Galilean observer  $S'$ . Instead of making a new

$$(16.222) \quad dx^2 + dy^2 = dx'^2 + dy'^2.$$

In fact, the quadratic expression  $dx^2 + dy^2$  is an *invariant*.

But in the space-time diagram

$$dt^2 + dx^2 \neq dt'^2 + dx'^2.$$

The invariant quantity is, by (16.205),

$$(16.223) \quad dt^2 - dx^2/c^2 = dt'^2 - dx'^2/c^2.$$

In geometry, we denote the invariant (16.222) by  $ds^2$  and call  $ds$  the *distance* between two adjacent points in the plane. This suggests that we should give a name to the square root of (16.223). However, the minus sign introduces a complication, since the invariant may be negative, and hence its square root imaginary. So we put

$$(16.224) \quad \epsilon ds^2 = dt^2 - dx^2/c^2,$$

where  $\epsilon = +1$  or  $-1$ , according as the expression on the right is positive or negative. We call  $ds$  the *separation* between the events  $(x, t)$  and  $(x + dx, t + dt)$ .

In the case of two events which are not adjacent, we define the separation as  $s = \int ds$ , taken along the straight line joining the points in the space-time diagram which correspond to them. Since  $dx/dt$  is constant along a straight line, we easily find

$$(16.225) \quad \epsilon s^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2/c^2,$$

where the two events in question are  $(x_1, t_1)$  and  $(x_2, t_2)$ . We note that, if the two events have the same  $x$ , the separation is simply the difference between the  $t$ 's; if they have the same  $t$ , the separation is the difference between the  $x$ 's, divided by  $c$ .

The lines in the space-time diagram satisfying one or other of the equations

$$(16.226) \quad dt = dx/c, \quad dt = -dx/c,$$

are called *null lines*, because the separation between any two points on such a line is zero. Clearly a null line represents the history of a flash of light traveling along the axis  $Ox$  of a Galilean frame, in one direction or the other.

So far we have made no hypothesis regarding the motion of a particle in a

$S'$ . In its history,  $A', A$  are two events, and the time interval  $T'$  between them (as judged by  $S'$ ) is the separation  $A'A$ . As judged by  $S$ , however, the interval between these events is  $T$ , the increase in  $t$  in passing from  $A'$  to  $A$ ; it is, in fact, the separation  $A'C$ , where  $A'C$  is drawn parallel to  $\Omega t$ . It is evident that  $T' \neq T$ .

There are some facts about the space-time diagram which we leave to the consideration of the reader. Why do the axes  $\Omega x't'$  not interlace with  $\Omega xt$ ? Why does  $T$  appear smaller than  $T'$  in Fig. 162, whereas we have proved the reverse in (6.221)? Is it true, for a triangle in the space-time diagram, that the sum of the separations represented by two sides is greater than the separation represented by the third side?

### 16.3. KINEMATICS AND DYNAMICS OF A PARTICLE

#### Composition of velocities.

Let  $P, Q$  be two particles traveling with uniform velocities  $u_1, u_2$  along the axis  $Ox$  of a Galilean frame of reference  $S$ . What is the relative velocity of the two particles? In Newtonian mechanics, we should answer:  $u_2 - u_1$ . In relativity, we say that this is only the *difference* between the velocities. The velocity of  $Q$  *relative* to  $P$  is the velocity of  $Q$  as estimated by a Galilean observer  $S'$  traveling along with  $P$ , i.e., using a Galilean frame of reference in which  $P$  is fixed.

Between  $S$  and  $S'$  we have the Lorentz transformation [cf. (16.213)]

$$(16.301) \quad x' = \gamma_1(x - u_1 t), \quad t' = \gamma_1 \left( t - \frac{u_1 x}{c^2} \right),$$

$$\gamma_1 = \frac{1}{\sqrt{1 - u_1^2/c^2}}.$$

Consider now the motion of  $Q$ ; for it,  $dx/dt = u_2$ , and its velocity as estimated by  $S'$  is

$$(16.302) \quad u' = \frac{dx'}{dt'} = \frac{dx - u_1 dt}{dt - u_1 dx/c^2} = \frac{u_2 - u_1}{1 - u_1 u_2/c^2}.$$

This is the law which replaces the Newtonian law,

$$(16.303) \quad u' = u_2 - u_1.$$

We note that, if  $u_1$  and  $u_2$  are small compared with  $c$ , (16.302) differs very

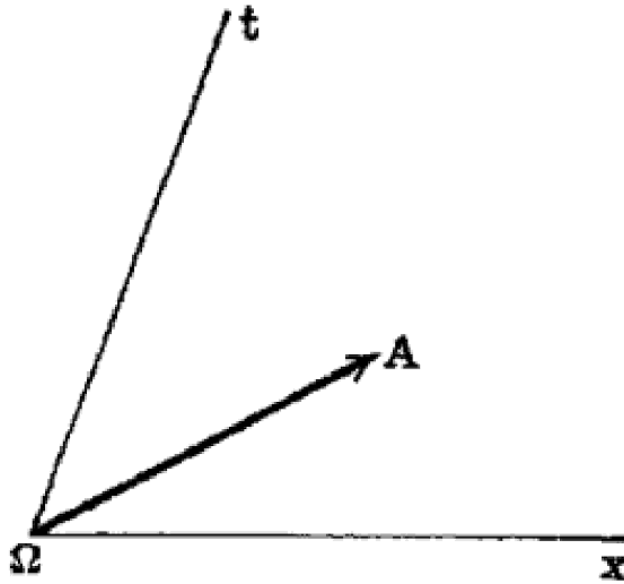


FIG. 164.—Space-time vector.

$$(16.310) \quad \xi' = \gamma(\xi - V\tau), \quad \tau' = \gamma\left(\tau - \frac{V\xi}{c^2}\right),$$

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

As we have stated, our problem is to build equations of motion invariant under a Lorentz transformation. The key to the solution is found in the idea of the space-time vector. *We shall form equations in which a space-time vector is equated to a space-time vector.*

Consider a particle moving along  $Ox$  with a general motion. This corresponds to some curve in space time. The proper time, measured from some initial point, may be taken as a parameter, and the equations of the curve written

$$x = x(s), \quad t = t(s).$$

Since  $ds$  is an invariant, the pair of quantities  $(dx/ds, dt/ds)$  is a space-time vector. We see this by differentiating (16.213). We call  $(dx/ds, dt/ds)$  the *absolute velocity* of a particle. Explicitly we have, by (16.305),

$$(16.311) \quad \frac{dx}{ds} = \gamma_u u, \quad \frac{dt}{ds} = \gamma_u,$$

$$\gamma_u^2 = 1 - u^2/c^2,$$

where  $u = dx/dt$ , the velocity of the particle relative to the Galilean frame of reference  $S$ , corresponding to  $Qxt$ . If the particle has a velocity small compared with that of light, so that  $u/c$  is small, the components of the absolute velocity are approximately  $(u, 1)$ .

Similarly,  $(d^2x/ds^2, d^2t/ds^2)$  is a space-time vector. We call it the *absolute acceleration*.

We now accept, as satisfactory from the point of view of invariance under the Lorentz transformation, the following *equations of motion*:

$$(16.312) \quad m_0 \frac{d^2x}{ds^2} = X, \quad m_0 \frac{d^2t}{ds^2} = T,$$

where  $m_0$  is a constant (the *proper mass* of the particle) and  $(X, T)$  is a space-time vector, called the *absolute force*.

In a different Galilean frame of reference  $S'$ , with velocity  $V$  relative to  $S$ , these equations read

$$m_0 \frac{d^2x'}{ds'^2} = X' \quad m_0 \frac{d^2t'}{ds'^2} = T',$$

where

$$(16.313) \quad X' = \gamma(X - VT), \quad T' = \gamma\left(T - \frac{VX}{c^2}\right),$$

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

This may be verified immediately by applying (16.213) to (16.312).

### Equations of motion in relative form.

Let us now put the equations of motion (16.312) into another form in order to show the relation of relativistic to Newtonian mechanics. Since  $ds = dt/\gamma_u$ , these equations may be written

$$(16.314) \quad \frac{d}{dt} (m_0 \gamma_u u) = X/\gamma_u, \quad \frac{d}{dt} (m_0 \gamma_u) = T/\gamma_u.$$

Let us define some terms, as follows:

$$(16.315) \quad \left\{ \begin{array}{l} \text{Relative mass} = m = m_0 \gamma_u = \frac{m_0}{\sqrt{1 - u^2/c^2}} \\ \text{Relative momentum} = mu. \\ \text{Relative force} = P = X/\gamma_u = X \sqrt{1 - u^2/c^2}. \\ \text{Relative energy} = E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}. \end{array} \right.$$

Then the first of (16.314) may be written

$$(16.316) \quad \frac{d}{dt} (mu) = P;$$

in words,

rate of change of relative momentum = relative force.

This is the *equation of motion of a particle* moving on the  $x$ -axis under the influence of a force  $P$ . It is of the Newtonian *form*, but with a remarkable difference. The (relative) mass of a particle is *not* a constant; it varies with the speed of the particle according to (16.315).

If  $u/c$  is small, the variation of  $m$  from the value  $m_0$  is insignificant, but on the other hand  $m$  tends to infinity as the speed of the particle ( $u$ ) tends to that of light ( $c$ ). No particle has ever been observed traveling with a speed equal to, or greater than, that of light. This physical fact agrees with the theory. If the speed of a particle were to increase up to and beyond the speed of light, the relative mass would become meaningless, passing through an infinite value to imaginary values.

To discuss the second equation of (16.314), let us first return to (16.224). This may be written

$$(16.317) \quad \left( \frac{dt}{ds} \right)^2 - \frac{1}{c^2} \left( \frac{dx}{ds} \right)^2 = 1,$$

since  $(dx/dt)^2 < c^2$  for a particle. Differentiation gives

$$(16.318) \quad \frac{dt}{ds} \frac{d^2 t}{ds^2} - \frac{1}{c^2} \frac{dx}{ds} \frac{d^2 x}{ds^2} = 0.$$

Thus, by (16.312),

$$(16.319) \quad T \frac{dt}{ds} - \frac{1}{c^2} X \frac{dx}{ds} = 0,$$

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2 \left(1 - \frac{u_1u_2}{c^2}\right),$$

$$u_3 = \frac{m_1\gamma_1u_1 + m_2\gamma_2u_2}{m_1\gamma_1 + m_2\gamma_2},$$

where  $\gamma_1^{-2} = 1 - u_1^2/c^2$ ,  $\gamma_2^{-2} = 1 - u_2^2/c^2$ .

10. A particle of proper mass  $m_0$  moves on the axis  $Ox$  of a Galilean frame of reference, and is attracted to the origin  $O$  by a (relative) force  $m_0k^2x$ . It performs oscillations of amplitude  $a$ . Show that the periodic time of this relativistic harmonic oscillator is

$$\tau = \frac{4}{c} \int_0^a \frac{f dx}{\sqrt{f^2 - 1}},$$

where

$$f = 1 + \frac{1}{2} \frac{k^2}{c^2} (a^2 - x^2).$$

Verify that, if  $c \rightarrow \infty$ ,  $\tau \rightarrow 2\pi/k$  (the Newtonian result); and show that if  $ka/c$  is small,

$$\tau = \frac{2\pi}{k} \left(1 + \frac{1}{8} \frac{k^2 a^2}{c^2}\right), \quad \text{approximately.}$$

\* This is the usual name, and not a very good one, for Galileo lived before Newton and of course had no idea of the theory of relativity. "Einstein frame of reference" would be a better name.

\* For an account of the Michelson-Morley experiment, see L. Silberstein, *The Theory of Relativity* (Macmillan Company, Ltd., London, 1924), p. 71.

\*  $3.00 \times 10^{10}$  cm. sec.<sup>-1</sup> or 186,000 mile sec.<sup>-1</sup>



## APPENDIX

### THE THEORY OF DIMENSIONS

Two physicists are shipwrecked on a desert island. After making qualitative observations of their surroundings, they wish to make measurements. But here a difficulty arises, for they have none of the usual apparatus of the laboratory—no meter scale, no set of weights, no clock.\* Everything they require they must construct for themselves.

If they can agree on the length of a certain stick as unit of length, the mass of a certain stone as unit of mass, and the duration of some simple repeatable experiment as unit of time, all will be well; both experimenters will assign the same *number* to the same measurable quantity. But why choose one stick rather than another, one stone rather than another, one experiment rather than another? If the two physicists are obstinate, each in his own preference of units, there is no valid argument by which one can persuade the other to yield.

This disagreement concerns only physical measurements. In the realm of pure mathematics, there is complete accord; both agree, for example, that

$$(1) \quad 2 + 2 = 4, \quad (x + 1)(x - 1) = x^2 - 1, \quad \int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

But in the matter of the choice of units, we may well imagine that neither physicist will yield to the other. So they decide to work independently, each constructing his own apparatus, measuring quantities in the units he prefers, and developing his own results. If they wish to discuss their work, how is one to interpret the results of the other? How far do their individual efforts contribute to the construction of a common science, independent of the choice of units? These are questions which belong to the theory of dimensions.

It may appear strange that we have been able to postpone to an appendix the discussion of these important questions. The explanation is that the theory of dimensions becomes necessary only when we wish to compare results for two different systems of units. In our work, we have used arbitrary units and letters (algebra) instead of actual numbers (arithmetic). Our results are valid quite

If  $Q_1$  and  $Q_2$  have the same dimensions, then

$$\left[ \frac{Q_1}{Q_2} \right] = [M^0 L^0 T^0],$$

and we say then that  $Q_1/Q_2$  is *dimensionless*. For example, the circular measure of an angle is obtained by dividing a length (arc) by a length (radius), and so an angle is dimensionless. It is easy to verify that the following combinations are also dimensionless:

$$\frac{\text{force} \times \text{time}}{\text{linear momentum}},$$

$$\frac{\text{force} \times \text{length}}{\text{energy}},$$

$$\frac{\text{moment of inertia} \times \text{angular velocity}}{\text{angular momentum}}.$$

*Exercise.* Einstein's radiation formula is  $E = h\nu$ , where  $E$  is the energy of a photon,  $\nu$  is its frequency, and  $h$  is Planck's constant. Show that Planck's constant has the dimensions of angular momentum.

## Change of units. First method.

Let us now consider two physicists  $S_1$  and  $S_2$ , who use different units of mass, length, and time. When they measure the same physical quantity, they record different results. But, as we shall now see, it is easy to pass from one numerical value to the other when we know the ratios of the two sets of units.

For symmetry, we introduce a third physicist  $S_0$ , using a third system of units; we shall call his units "absolute" for purposes of reference, without meaning to imply that they are in any way more fundamental than the units of  $S_1$  or  $S_2$ . Let the units of  $S_1$  contain  $m_1$ ,  $l_1$  and  $t_1$ , absolute units of mass, length, and time, respectively; and let the units of  $S_2$  contain  $m_2$ ,  $l_2$ , and  $t_2$  absolute units.

Consider a physical quantity  $Q$  with dimensions  $[M^\alpha L^\beta T^\gamma]$ . This quantity is measured by  $S_0$ ,  $S_1$ , and  $S_2$ , with numerical results as follows:

$S_0$	$S_1$	$S_2$
$Q_0$	$Q_1$	$Q_2$

Now every unit of mass recorded by  $S_1$  corresponds to  $m_1$  absolute units, and similarly for length and time. Hence, one  $S_1$ -unit of the quantity measured

describe is that in common use.

Compare the equations (1) with the following:

$$(9) \quad 1 \text{ meter} = 100 \text{ cm.}, \quad 1 \text{ lb.} = 453.6 \text{ gm.}, \\ 22 \text{ ft. per sec.} = 15 \text{ miles per hr.}$$

These are true statements, but they differ from (1) in an important respect: the equations (1) involve only pure numbers, whereas (9) involve measurable physical quantities. To distinguish them, we may call (1) mathematicians' equations (or briefly *M-equations*) and (9) physicists' equations (or *P-equations*). We know what we can do with M-equations according to the methods of algebra and calculus. There are certain rules of manipulation, which we apply with confidence that we shall never reach a false conclusion. Let us boldly apply the rules of algebraic manipulation to P-equations, treating such words as meter, cm., lb. as if they were *ordinary algebraic symbols*. A word of warning, however—the signs =, +, and – are to be used to connect only quantities of the same type, i.e., of the same dimensions.

We think again of two physicists  $S_1$  and  $S_2$ . Let  $S_1$  name his units  $M_1, L_1, T_1$ ; and let  $S_2$  name his units  $M_2, L_2, T_2$ . These are *names* (like gm. or cm.), not numbers. If  $S_1$  measures a length, he records the result in the form

$$Q = Q_1 L_1;$$

this is a P-equation, in which  $Q$  stands for “the quantity which is measured,” and  $Q_1$  is a number. More generally, if  $S_1$  measures a quantity with dimensions  $[M^\alpha L^\beta T^\gamma]$ , he records

$$(10) \quad Q = Q_1 M_1^\alpha L_1^\beta T_1^\gamma,$$

where  $Q_1$  is a number. If  $S_2$  measures the same quantity, he records

$$(11) \quad Q = Q_2 M_2^\alpha L_2^\beta T_2^\gamma.$$

There is nothing novel about this; it is what we do when we write

$$\text{acceleration due to gravity} = 32 \text{ ft. sec.}^{-2}$$

$$\text{acceleration due to gravity} = 980 \text{ cm. sec.}^{-2}$$

Now we bring into operation our assumption that the P-equations (10) and (11) may be treated in the same way as we should treat M-equations. We get at once

$$(12) \quad Q_1 M_1^\alpha L_1^\beta T_1^\gamma = Q_2 M_2^\alpha L_2^\beta T_2^\gamma,$$

and

$$(13) \quad \frac{Q_2}{Q_1} = \left(\frac{M_1}{M_2}\right)^\alpha \left(\frac{L_1}{L_2}\right)^\beta \left(\frac{T_1}{T_2}\right)^\gamma.$$

If we interpret  $M_1/M_2$  to mean the ratio of the unit  $M_1$  to the unit  $M_2$ , then  $M_1/M_2$  is a pure number—in fact, the measure of  $M_1$  in terms of  $M_2$ . Since  $L_1/L_2$  and  $T_1/T_2$  may also be regarded as pure numbers, (13) is an M-equation, although (12) (from which it was obtained) is a P-equation.

Equation (13) is what we have been seeking—a formula to give  $Q_2$  when  $Q_1$  and the ratios of the units are known. If we lack confidence in it, because it has been obtained by a symbolic method, we can reassure ourselves by turning back to the first method; there only M-equations were used, and the deduction of (6) and (7) is logically sound. We see that (12) is merely the P-equation corresponding to the M-equation (6), and (13) is the same as (7)—both M-equations.

When we actually carry out a conversion from one system of units to another, it is the P-equation (12) rather than the M-equation (13) that we use. It would, however, be more correct to say that we use neither. Therein lies the simplicity of the symbolic method; we treat each problem on its merits, without having to remember anything, except that it is permissible to use the symbolic method, in which words are treated as algebraic symbols. The formulas (12) and (13) were obtained only for purposes of comparison with (6) and (7).

To show the symbolic method in action, let us convert an acceleration of 32 ft. sec.<sup>-2</sup> into mile hr.<sup>-2</sup>\* First we write down

$$1 \text{ mile} = 5280 \text{ ft.}, \quad 1 \text{ hr.} = 3600 \text{ sec.},$$

so that

$$1 \text{ ft.} = \frac{1}{5280} \text{ mile}, \quad 1 \text{ sec.} = \frac{1}{3600} \text{ hr.}$$

Then,

Apart from its use in the change of units, the theory of dimensions has three important applications:

(i) It supplies us with a useful check against slips in calculation.

(ii) It suggests forms of physical laws.

(iii) It enables us to predict the behavior of a full-scale system from the behavior of a model.

These applications will now be explained.

(i) Provided that we do not insert numerical values, the dimensions of every combination of symbols occurring in our work are obvious. For example, if  $a$  is the length of a pendulum and  $g$  the acceleration due to gravity, then

$$\left[ \frac{a}{g} \right] = [T^2].$$

The basic law of motion (1.402) is dimensionally homogeneous, in the sense that both sides have the same dimensions, viz.,  $[MLT^{-2}]$ . The operations we perform on this equation may change the dimensions of the two sides, but they are both changed in the same way. Thus, at all stages of our deductions we have dimensionally homogeneous equations. Indeed, it is inevitable that this should be so, since otherwise the two sides of an equation would change differently on change of units, and if true for one system of units would not be true for another. This gives a useful check. For example, suppose we are engaged in working out the formula for the periodic time of small oscillations of a simple pendulum. As a result of our work we arrive, perhaps, at the result

$$\tau = 2\pi \frac{a}{g}.$$

Dimensionally, this reads

$$[T] = [T^2],$$

which shows that the result is incorrect. Such a check will, of course, never be of any assistance as far as a numerical coefficient is concerned; for example, the theory of dimensions alone cannot tell us that

$$\tau = \pi \sqrt{\frac{a}{g}}$$

is incorrect.

*Exercise.* It is suggested that the equation of motion of a particle on a line is

Hence the full-scale lift is

$$Y = Y' \frac{\rho U^2 l^2}{\rho' U'^2 l'^2}.$$

If the density of the air is the same for both cases, this becomes

$$Y = Y' \frac{U^2 l^2}{U'^2 l'^2}.$$

When we insert the numerical values for  $Y'$ ,  $U'$ ,  $l'$ , obtained from experiment on a model in a wind tunnel, and the values of  $U$  and  $l$  appropriate to the full-scale wing in flight, we are able to read off the value of the lift  $Y$ .

*Exercise.* In order to study the deflections in an elastic beam with continuous and isolated loads (cf. Sec. 3.3), an engineer builds a model of the same material with a linear ratio 1:100. Show that, if the deflections in the model are to be one one-hundredth of the full-scale deflections, the continuous load per unit length in the model-must be one one-hundredth of the full-scale load. In what ratio should the isolated loads be reduced?

\* We may suppose the sky perpetually overcast, so that the rotation of the heavens cannot be used as a clock.

\* It is convenient to write each unit in the singular, to avoid waste of energy in deciding whether to use the singular or the plural. This is a mathematical symbolism, and in it simplicity is more important than grammar.



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